Universitat Politècnica de Catalunya
Facultat de Matemàtiques i Estadística

Degree in Mathematics

Bachelor's Degree Thesis

Shannon capacity and the Lovász Theta function

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May, 2019

Abstract

This thesis focuses on the study of two graph parameters known as the Shannon capacity and the Lovász number. The first was introduced by Claude Elwood Shannon on [1] and describes the maximum rate at which information can be transmitted through a noisy channel of comunication, where the noise of the channel is encoded by a graph. The second one was introduced by László Lovász as an attempt to determine the first. We start by introducing all the required concepts in order to formally state the problem by Shannon. We then move on to develop some theory regarding the spectrum of a set of matrices associated with graphs, providing an algebraic aproach to the matter in question. Then we prove some results due to Lovász on [2] that allow us to effectively compute the Shannon capacity for specific families of graphs, including the Kneser graphs and the perfect graphs. We end up by studying further properties of the Lovász number, of particular interest for the case of perfect graphs.

Keywords

Shannon capacity, Lovász number, spectral bounds for graphs, Kneser graphs, Kneser spectrum, perfect graphs, weak perfect graph theorem

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1. Graph Theory and Shannon Capacity

Introduction

Graph theory is a relatively new but very broad branch of mathematics, hidden in many of our daily quotidian lives. The story of the seven bridges of Königsberg has been told thousands of times, and it goes something like this: The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River and was composed by four main bodies of land, only connected to each other by seven bridges (see figure 1). The problem was to devise a walk through the city such that each bridge was crossed once and only once.

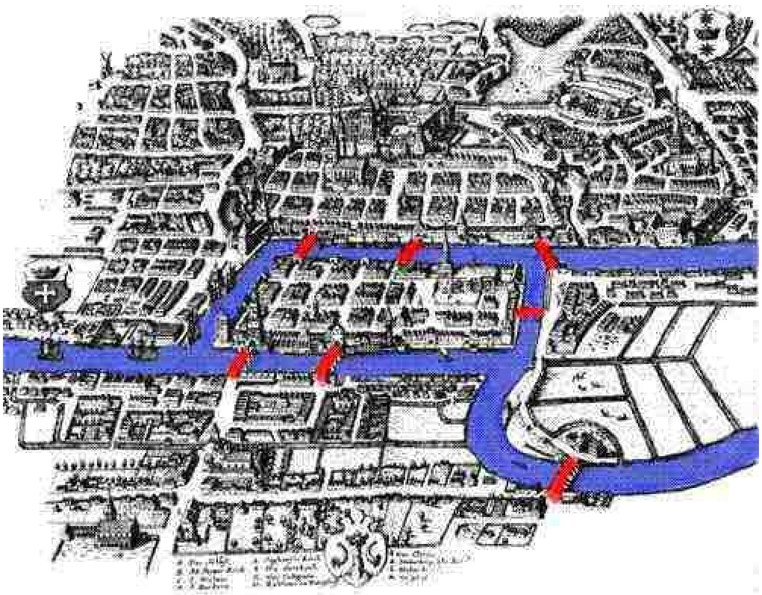


Figure 1: Map of Könisberg back in the 1700's showing the four body lands conected by the seven bridges (in red)

After a few attempts, one might convince himself that such a walk is impossible, but why? Is it because we're not clever enough to figure it out, or perhaps there is something bigger going on? In 1736 Lehonard Euler [3] came into rescue, not only proving the problem negatively but also founding the field of graph theory along the way.

The first thing Euler did was to remove all the irrelevant information for the problem in order to reformulate it in abstract therms. He denoted the land masses by "vertices" and the bridges connecting them by "edges" between those vertices, creating what now is known as a "graph". The problem then becomes the following: Choose a vertex from where to start, then traverse along an edge from this vertex to another vertex, and again, and so on, until every edge has been traversed once and only once.

Now he reasoned as follows. Start by giving a letter to each vertex, and for any walk on the graph, construct a word by writting down each vertex we visit. For instance, word *ABA* means that we start on vertex *A*, then move to vertex *B* and then move back to *A*. Now suppose that there is a walk traversing each edge once and only once. Then the word formed after this walk has to have a length equal to the

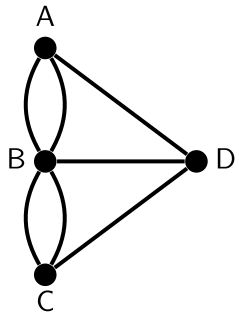


Figure 2: Könisberg's graph representation

number of edges on the graph plus one, 8 in our case.

On the other hand, consider how many times each letter must appear in the "body" of that word (that is, not in the first place nor the last). For any letter appearing in the body part, that means that we entered a vertex and we left it right after. For instance vertices A, D, and C can only be entered-leaved once since they have only 3 edges incident on them, so they must appear once. Similarly vertex B can't appear more than twice because it only has 5 edges incident on it. Therefore the body of the word can not have more than 1+1+1+2=5 letters, and if we add the first and the last vertices, we end up with a word of length 7 at most. But we previously concluded that such a word had to have length 8! This contradiction ensures that a walk traversing each edge once and only once it is impossible in the case of the Könisberg graph.

Euler actually showed that in general for any graph that admits a walk traversing each edge once an only once (now called *Eulerian* graphs), then there can not be more than 2 vertices with an odd number of edges incident to it.

1.1 Basics of Graphs

Definition 1.1 (Graph)**.** A *simple graph* (or just *graph*) is a pair of sets (V, E) where V is the nonempty *set of vertices* and E is the *set of edges*, consisting of distinct unordered pairs of distinct elements in V.

Formally, given a graph $G = (V, E)$, we have $E \subseteq \{\{u, v\} \mid u, v \in V\}$. We will that say u and v are *adjacent* if there is an edge connecting u and v, that is $\{u, v\} \in E$.

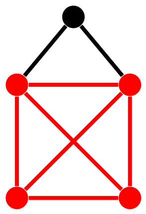
Diferent restrictions over the edge set E give diferent definitions of graphs. For instance, if we allow repetitions on E we have a *multigraph*. If instead we consider the edges to be ordered pairs of vertices we get a *directed graph*. We could also allow edges to connect a vertex to itself, which would give us a *graph with loops*. Note that all fesaible combinations of such restrictions (and possibly more) could coexist, giving rise to lots of diferent definitions for a graph. As much interesting and fancy this graphs might look like, throughout this document we will only consider graphs satisfying the definition 1.1.

Definition 1.2 (Complementary graph)**.** The *complementary graph* $\overline{G}$ of a graph $G = (V, E)$ is a graph with vertex set V and where two vertices are adjacent in $\overline{G}$ if they are not adjacent in G.

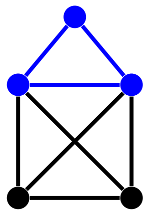
For example, if K_n denotes the *complete graph on n vertices* (where every pair of vertices are adjacent) then $\overline{K_n}$ is the *null graph* where none vertices are adjacent.

Definition 1.3 (Subgraph and induced subgraph)**.** Given a graph $G = (V, E)$, a *subgraph* $G' = (V', E')$ of G is a graph such that $V' \subseteq V$ and $E' \subseteq E$. An *induced subraph* of G is a subgraph G' satisfying also that two vertices v_1 and v_2 of G' are adjacent if and only if they also are adjacent in G.

Definition 1.4 (Clique)**.** A *clique* Q of a graph G is an induced subraph that is also a complete graph. We will call the number of vertices of a clique its *size*.



(a) clique of size 4



(b) clique of size 3

Figure 3: Envelop graph

The size of the largest clique a graph G contains is denoted by $\omega(G)$.

In the envelop graph shown in Figure 3, the size of its largest clique is 4. The size of the largest clique in a K_n is trivially $\omega(K_n) = n$. For small graphs, finding $\omega(G)$ may be easy, but in the general case no known polynomial time algorithms are known.

Lets move on to the counterpart of a clique.

Definition 1.5 (Independent set)**.** An *independent set* S of a graph G is a subset $S \subset V$ of the vertex set of G such that non vertices in S are adjacent.

The size of the largest independent set of a graph G is denoted by $\alpha(G)$.

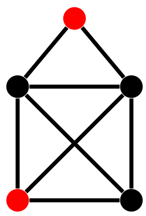


Figure 4: Independent set of size 2. No independent set of size 3 exists.

An independent set S of a graph G can also be seen as an induced subgraph which is totally disconected (the edge set is empty). By considering the complementary graph then $\overline{S}$ is a complete subgraph (every vertex is adjacent to each other) of $\overline{G}$, that is, $\overline{S}$ is a clique of $\overline{G}$. Moreover, if S is an independent set of maximum size of G, then $\overline{S}$ is a clique of maximum size of $\overline{G}$. This gives us the following relation:

$$\alpha(G) = \omega(\overline{G}) \tag{1.1}$$

Since it is trivial to compute the complementary of a graph, we see that it is equally hard to calculate either $\alpha(G)$ or $\omega(G)$.

Now we will introduce two other graph parameters we will need later on.

Definition 1.6 (Proper colouring and chromatic number)**.** A *proper k-colouring* of a graph $G = (V, E)$ is a map $f : V \to \{1, \dots , k\}$ such that $f(u) \neq f(v)$ whenever u and v are adjacent. The minimum k for which G has a proper k-colouring is called the *chromatic number of G*, and is denoted by $\chi(G)$.

If Q is a clique in G, then a proper colouring of G has to colour each vertex in Q with a different colour since all vertices are pairwise adjacent, so at least $\omega(G)$ colours are needed to colour G. That is:

$$\omega(G) \leqslant \chi(G). \tag{1.2}$$

(a) maximum clique of size 2 (b) 3-colouring

Figure 5: Graph with a chromatic number strictly greater than a clique number

Definition 1.7 (Clique covering and clique cover number)**.** A *clique cover* of a graph G is a partition of the vertices of G into cliques. The minimum number of cliques needed to cover a graph G is called the *clique cover number* of G and is denoted by $\overline{\chi}(G)$.

Similarly to how the counter part of a clique is an independent set, the counterpart of a colouring is actually a clique cover in the following sense: if we colour a graph G with k colours, then each colour class is an independent set of G, and thus each colour class is also a clique of the complement graph $\overline{G}$. Therefore we have

$$\chi(G) = \overline{\chi}(\overline{G}). \tag{1.3}$$

Moreover, if we have a clique cover of a graph G then the vertices of any independent set of G have to belong to diferent cliqes (since otherwise they would be adjacent), giving the following inequality analog to 1.2:

$$\alpha(G) \leqslant \overline{\chi}(G). \tag{1.4}$$

A real world problem that requires the chromatic number of a graph to be computed is the following. Suppose we are a university and we have a list of n different lectures $\{v_1, v_2, \dots , v_n\}$ and each one is scheduled during the time interval $[a_i, b_i]$. Supose also that we have a bunch of smart and unfatigable professors that can give any of the n lectures. The question then is: what is the minimum number of professors needed in order for all the lectures $\{v_1, \dots , v_n\}$ to be taught, given that a professor cannot give two lectures whose time intervals coincide?

(a) a maximum independent set of size 2 (b) a cover with 3 cliques

Figure 6: Graph with a clique covering number strictly greater than independency number

If we let a graph G have as vertex set the diferent lectures to be taught, $V = \{v_1, v_2, \dots, v_n\}$, and edges between pairs of lectures that coincide in time, $E = \{v_i v_j \mid [a_i, b_i] \cap [a_j, b_j] \neq \varnothing\}$, then we have that an assignment of professors (colours) to lectures (vertices) results in a proper colouring of G. The minimum of such professors is then $\chi(G)$.

Lets try to make such an assignement. Without loss of generality we sort the lectures by the time when they start and we write $v_i \leqslant v_j$ if v_i starts before v_j. Now proceed as follows: start by assigning to the first lecture, v_1, the colour 1. Then, in order, assign to any given v the minimum positive integer not assigned to any of the previous v' *adjacent* to v. Once we assigned an integer to the last vertex, we end up with a k-colouring, being k the largest integeer ever used.

In the practical example of the lectures and teachers, the algorithm described above results first in puting the teachers on a line. Then, through the day, whenever a lecture has to begin, take the teacher on the head of the line and assign it to that lecture, and as soon as that teacher finishes giving the lecture, ask him to go back to the head of the line.

Clearly our method works. But does this method minimize the number of teachers needed to teach the lectures? In other terms, does the k-colouring on G described above satisfy $k = \chi(G)$? Yes it does. Suppose vertex v_i is assigned the greatest colour class k. Then the reason it is coloured by k is becasue there must be $j_1, j_2 \dots j_{k-1}$ (all less than i) such that each v_{j_r} is coloured by r and also $[a_{j_r}, b_{j_r}] \cap [a_i, b_i] \neq \emptyset$, for each r with $0 \leqslant r \leqslant k-1$. However, since $j_r < i$ then $a_{j_r} \leqslant a_i$ and therefore all intervals $[a_{j_r}, b_{j_r}]$ do contain a_i. Hence $\{v_{j_1}, \dots, _{j_{k-1}}, v_i\}$ form a clique of size k, and from the definition of $\omega(G)$ and inequality (1.2) we have that:

$$k \leqslant \omega(G) \leqslant \chi(G) \leqslant k$$

and therefore $\omega(G) = \chi(G) = k$.

The previous argument motivates the following definition:

Definition 1.8 (Perfect graph)**.** A graph G is a *perfect graph* if $\omega(H) = \chi(H)$ for every induced subgraph $H \subset G$.

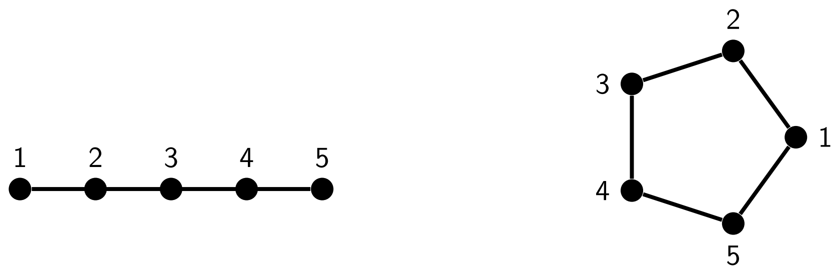
The graph considered in teachers and lectures problem was actually a perfect graph, and as we will see in greater detail on Section 5, perfect graphs are as good as it gets in the vast pool of graphs.

1.2 Introduction to the Shannon's capacity

In 1956, the engineer and mathematician C. E. Shannon [1] studied zero-fault information transmission. More specificely, he wanted to know which was the highest rate at which information could be transmitted through a noisy channel of comunication.

For instance, suppose we have a channel that supports the transmition of 5 'letters' per unit of time, lets say those are the natural numbers 1, 2, 3, 4 and 5. Suppose also that there is some noise on the channel that modifies the information transmitted in the following way: whenever we send a number $s = 1, \dots, 5$ through one end of the channel, the other end recives the number $s \pm \epsilon$, with $\epsilon \in (0, 1)$. Because of this noise, sending a 1 might produce the same result as sending a 2, so 1 and 2 are confusable, as well as 2 and 3, etc.

We could represent the letters supported by the channel as the vertices of a graph, and the fact that two symbols may be confused by an edge joinin those vertices. In this way, for the previous example we get the *path graph* on 5 vertices P_5. If it is the case that the channel also confuses 1 and 5 (maybe because we consider the numbers to be integers modulo 5), the graph representing the channel becomes the *cycle graph* on 5 vertices C_5.



(a) Path graph P_5 (b) Cycle graph C_5

In adition to comfusable letters, we may say that two words of length n are confusable if in every position of each word we have a pair equal letters or a pair of confusable letters. For example, if C_5 represents our channel then the words (12) and (51) are undistinguishable since 1 and 5 are confusable, and so are 2 and 5. The ultimate goal is to develop a strategy which allows an optimal comunication through the given channel encoded in a graph. To state formally what we mean by a "strategy" being "optimal" we need the following definition:

Definition 1.9 (Strong product of graphs)**.** Let G and H be two graphs on the vertices $V(G)$ and $V(H)$ respectively. The *strong product of G and H* is the graph $G \boxtimes H$ on vertices $V(G) \times V(H)$ and where (x_1, y_1) and (x_2, y_2) are adjacent in $G \boxtimes H$ whenever x_1 and x_2 are equal or adjacent and also y_1 and y_2 are equal or adjacent.

Note that vertices $(x_1, y_1), (x_2, y_2)$ are adjacent in $G \boxtimes H$ if and only if vertices $(y_1, x_1), (y_2, x_2)$ are adjacent in $H \boxtimes G$, implying that both $G \boxtimes H$ and $H \boxtimes G$ are actually the same graph. This allows us to define $G^n := G \boxtimes G \boxtimes \dots \boxtimes G$ (n times). Then the vertices of G^n correspond to words of length n, and two words of length n are confusable if they are adjacent in G^n.

We are now ready to rigorously define what Shannon was actually studing.

Definition 1.10 (Shannon capacity)**.** Let G be a graph with independency number $\alpha(G)$. The *Shannon capacity* of G is defined as:

$$\Theta(G) := \lim_{n\to\infty} \alpha(G^n)^{\frac{1}{n}} = \sup_n \alpha(G^n)^{\frac{1}{n}}.$$

The meaning of the Shannon capacity of a graph is then the limit, as the length of the words increases, of the number of letters that can be sent per unit of time through the channel encoded by the graph without risk of confusion. To show that this limit exists and is actually the supremum we will use a version of Fekete's lemma. We need an obervation first:

Lemma 1.11. *Let G and H be two graphs. Then*

$$\alpha(G)\alpha(H) \leqslant \alpha(G \boxtimes H)$$

Proof. Let S be a maximum independent set of vertices in G, and let T be a maximum independent set of vertices in H, with $|S| = \alpha(G)$ and $|T| = \alpha(H)$. Then the set $S \times T$ is a subset of the vertex set of $G \boxtimes H$ of cardinality $\alpha(G)\alpha(H)$. Any two vertices on $S \times T$, say $(x_1, y_1), (x_2, y_2)$ are clearly non-adjacent in $G \boxtimes H$ since neither x_1, x_2 nor y_1, y_2 are adjacent. Therefore $S \times T$ is an independent set on the product and thus $|S \times T| \leqslant \alpha(G \boxtimes H)$. ■

The key lemma is the following.

Lemma 1.12 (Fekete's lemma)**.** *Let $f : \mathbf{N} \to \mathbf{N}$ be a function for which $f(m+n) \geqslant f(m) \cdot f(n)$ for all $m, n \in \mathbf{N}$. Then $\lim_{n\to\infty} f(n)^{1/n}$ exists (possibly ∞), and equals $\sup_n f(n)^{1/n}$.*

Proof. Let $n = mq + r$ for some positive integeers m, q, r greater or equal than one. From the hypothesis, f satisfies inductively $f(mq + r) \geqslant f(m)^q \cdot f(r)$. By fixing r and m then

$$\begin{aligned}
\liminf_{q\to\infty} f(mq+r)^{\frac{1}{mq+r}} &\geqslant \liminf_{q\to\infty} \left[f(m)^q \cdot f(r)\right]^{\frac{1}{mq+r}} \\
&= \liminf_{q\to\infty} \left[f(m)^{\frac{q}{mq+r}} \cdot f(r)^{\frac{1}{mq+r}}\right] \\
&= \liminf_{q\to\infty} \left[\left(f(m)^{\frac{1}{m}}\right)^{\frac{mq}{mq+r}} \cdot f(r)^{\frac{1}{mq+r}}\right] \\
&= f(m)^{\frac{1}{m}}.
\end{aligned}$$

Thus, for any given m:

$$\liminf_{n\to\infty} f(n)^{\frac{1}{n}} = \inf_{0\leqslant r\leqslant q} \liminf_{q\to\infty} f(mq+r)^{\frac{1}{qm+r}} \geqslant f(m)^{\frac{1}{m}},$$

and finally letting $m \to \infty$ the claim follows. ■

Now for any graph G we take $f(n) := \alpha(G^n)$. By lemma 3.5 clearly $f(n)$ satisfies $f(n+m) \geqslant f(m)f(n)$ and thus $\lim \alpha(G^n)^{1/n}$ exists and equals the supremum of definition 1.10.

Turning back to the example when $G = C_5$, we could form 2^n uncomfusable words of length n by using only the letters $\{1, 4\}$, so $\Theta(C_5) \geqslant 2$. However, a simple check shows that the set of vertices $\{13, 21, 34, 42, 55\}$ is an independent set in C_5^2 and therefore, for even n ,we can actually form $5^{n/2}$ uncomfusable words of length n by constructing words using only those pairs of letters. This does better than using only two letters since $2^n < 5^{n/2}$, implying then that $\Theta(C_5) \geqslant \sqrt{5}$.

Finally, for an example where $\Theta(G)$ can be computed in a straightforward manner, let G be a graph encoding a channel that supports yet again 5 letters, but where letters $\{1, 2, 3\}$ are all comfusable to each other, and so do letters $\{3, 4, 5\}$. The graph arising from this setup is often called the *bowtie graph*. We let

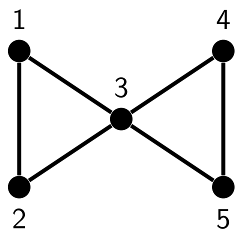


Figure 8: Bowtie graph or butterfly graph

$V = \{1, \dots, 5\}$ be the vertex set of G, and we let Q_1, Q_2 be the induced subgraphs of G induced by vertices $\{1, 2, 3\}$ and $\{3, 4, 5\}$ respectively. Then each Q is a triangle of G (and thus a clique). Now consider the product G^n. For any two non-adjacent vertices of G^n, say $(x_1, \dots x_n)$ and $(y_1, \dots, y_n)$, there must be an i such that the i-th components x_i, y_i have to belong to distinct $Q's$, since otherwise the vertices would be adjacent. Therefore any independent set S of G^n can be constructed by building vertices by choosing each of its component from either Q_1 or Q_2, so it must be $|S| \leqslant 2^n$. This implies that $\alpha(G^n) \leqslant 2^n$, but since $\alpha(G^n) \geqslant \alpha(G)^n$ and clearly $\alpha(G) = 2$ we conclude that $\alpha(G^n) = \alpha(G)^n$. Hence

$$\Theta(G) = \lim_{n\to\infty} \alpha(G^n)^{1/n} = \lim_{n\to\infty} \left(\alpha(G)^n\right)^{1/n} = \alpha(G) = 2.$$

The crucial property that allowed us to prove $\alpha(G^n) = \alpha(G)^n$ was the fact that the bowtie graph G could be covered with two cliques. This reasoning can be generalized and results into the general Theorem stated by Shannon:

Theorem 1.13 (Shannon's Theorem)**.** *If G is a graph which can be covered by $\alpha(G)$ cliques, that is G satisfies $\alpha(G) = \overline{\chi}(G)$, then*

$$\Theta(G) = \alpha(G).$$

■

As we will see on section 5, all perfect graphs exhibit this property, and the bowtie graph is one of them. However, computing the Shannon capacity for graphs not covered by this result (such as the pentagon C_5) is hard and more advanced techniques have to be developed, as did Lovász on [2] and as we expose in the following sections.

2. Spectral Graph Theory

2.1 Notions of Linear Algebra

Throughout the following sections, I will denote the identity matrix and J the matrix containing all 1's. Vectors will be taken in column form, and we shall write $\mathbf{j}$ for the vector consisting of all $1's$. The ambient space will always be the usual euclidean space $\mathbf{R}^d$ and unless stated, the dimension will always be clear from the context.

We proceed by recalling some definitions and results from basic linear algebra which will be used later. The proof of this results can be found on any book covering a first course on Linear Algebra.

Theorem 2.1 (Spectral Theorem)**.** *Suppose A is a symmetric matrix with entries on the real numbers. Then*

(i) Every eigenvalue λ of A is a real number.

(ii) There is a set $\{\mathbf{u_1}, \dots, \mathbf{u}_n\}$ which is a base of the eigenvectors of A and is an orthonormal basis of $\mathbf{R}^n$.

(iii) $A = UDU^{\mathsf{T}}$, where D is a diagonal matrix formed by the eigenvalues λ_i of A and U is an orthonormal matrix with columns the distinct eigenvectors $\mathbf{u}_i$.

Definition 2.2. A $n \times n$ symmetric matrix M is said to be *positive semidefinite* (resp. *negative semidefinite*) if $\mathbf{x}^{\mathsf{T}} M\mathbf{x} \geqslant 0$ (resp. $\mathbf{x}^{\mathsf{T}} M\mathbf{x} \leqslant 0$) for all $\mathbf{x} \in \mathbf{R}^n$.

An easy comprovation shows that that the following characterization of the definitenes of a matrix holds:

Proposition 2.3. *A symmetric matrix M is positive semidefinite (resp. negative semidefinite) if and only if all its eigenvalues are non-negative (resp. non-positive).*

Proposition 2.4. *Let A be a symetric matrix wich is also positive semidefinite. Then there is a matrix V such that*

$$A = V^{\mathsf{T}} V.$$

Proof. Since A is symmetric, by Theorem 2.1 we can write $A = U^{\mathsf{T}} DU$ where D is a diagonal matrix with entries equal to the eigenvalues of A.

Now using the fact that A is positive semidefinite, by proposition 2.3 we know that the diagonal of D consists of non-negative elements so by taking square roots:

$$A = U^{\mathsf{T}} DU = U^{\mathsf{T}} D^{\frac{1}{2}} D^{\frac{1}{2}} U = (D^{\frac{1}{2}} U)^{\mathsf{T}} (D^{\frac{1}{2}} U)$$

and letting $V := (D^{\frac{1}{2}} U)$ the claim follows. ∎

2.2 Spectrum of a Graph

Spectral theory is the study of the *spectrum*, the set of eigenvectors, and its related eigenvalues of some matrices. In our case, given a graph G we will construct a matrix which encodes the information contained in G, and by investigating the properties of this matrix from an algebraic point of view we will deduce some combinatorial results about G.

Definition 2.5 (Adjacency matrix)**.** Given a graph G on the vertex set $V = \{1, \dots, n\}$, the *adjacency matrix* of G is a $n \times n$ matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if there is an edge between i, j and $a_{ij} = 0$ otherwise.

As said before, the adjacency matrix of a graph actually encodes the adjacency relations of its vertex in the following way: let A be the adjacency matrix of a graph G on vertices $\{1, \dots, n\}$ and let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the canonical basis on $\mathbf{R}^n$. Then:

$$\mathbf{e}_j^\top A \mathbf{e}_i = 1 \iff i, j \text{ are adjacent.}$$

If G is an undirected graph, the adjacency matrix $A(G)$ is symmetric and thus by the spectral Theorem has an orthonormal basis of eigenvectors corresponding to real eigenvalues. This leads to the following definition:

Definition 2.6 (Spectrum of a Graph)**.** If G is a simple graph, we define the *spectrum* of G, spec(G) to be the multiset consisting of all the eigenvalues of the adjacency matrix $A(G)$.

Here is a list of the spectra of some common graphs:

Graph	Spectrum
K_5	$4, -1, -1, -1, -1$
$K_{3,3}$	$3, 0, 0, 0, 0, -3$
C_5	$2, \frac{1}{2}(-1+\sqrt{5}), \frac{1}{2}(-1+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5}), \frac{1}{2}(-1-\sqrt{5})$
Petersen graph	$3, 1, 1, 1, 1, 1, -2, -2, -2, -2$

The spectrum of the complete graph K_n can easily be computed as follows. Let A be the adjacency matrix of K_n, which satisfies $A = J - I$. The rank of J is 1 so there is one nonzero eigenvalue equal to n (with $\mathbf{j}$ as eigenvector). All the remaining eigenvalues of J are 0. Subtracting the identity shifts all eigenvalues by -1 since

$$A\mathbf{v} = (J - I)\mathbf{v} = J\mathbf{v} - \mathbf{v}.$$

Thus the eigenvalues of K_n are $n-1$ and -1 (of multiplicity $n-1$).

The spectra of another family of graphs, namely the *cycle graphs*, is of particular interest for this thesis and will also be computed. Let C_n be the cycle graph on vertices $\{1, \dots, n\}$, where i and j are adjacent if and only if $|i-j| \equiv 1 \pmod{n}$. Let A be the adjacency matrix of C_n and let $\mathbf{v} = (1, \zeta, \zeta^2, \dots, \zeta^{n-1})$ where ζ is a n-th root of the unity. Then

$$A\mathbf{v} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & & & 0 \\ 0 & 1 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 1 & 0 \\ 0 & & & 1 & 0 & 1 \\ 1 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \vdots \\ \zeta^{n-1} \end{pmatrix} = \begin{pmatrix} \zeta + \zeta^{n-1} \\ \zeta^2 + 1 \\ \zeta^3 + \zeta \\ \vdots \\ \vdots \\ \zeta^n + \zeta^{n-2} \end{pmatrix} = (\zeta + \zeta^{-1})\mathbf{v}$$

so $\zeta + \zeta^{-1}$ is an eigenvalue of A for each ζ_k such that $\zeta_k^n = 1$. By writting $\zeta_k = e^{\frac{2\pi ik}{n}}$ and using $e^{xi} + e^{-xi} = 2\cos x$, we conclude that

$$\mathrm{spec}(C_n) = \left\{2\cos\left(\frac{2\pi k}{n}\right) : 0 \leqslant k < n\right\}. \tag{2.1}$$

Note that the largest eigenvalue is $\lambda_{max} = 2$, while the smallest one depends on the parity of n: if n is even then $\lambda_{min} = -2$, and if n is odd then $k = \frac{n-1}{2}$ gives $\lambda_{min} = 2\cos\left(\frac{(n-1)\pi}{n}\right) = -2\cos\left(\frac{\pi}{n}\right)$.

2.3 Spectral bound of the independence number

Now we will present some results where a graph parameter is related to the spectrum of a graph. The first of them is of great relevance and is due to Hoffman.

Definition 2.7 (Regular graph)**.** Let G be a graph and let v be a vertex of G. The *degree* (or *valency*) of v is the number of vertices adjacent to v. A graph G in such that every vertex a has degree d is called a *regular graph of degree d* or also d-regular graph.

Theorem 2.8 (Hoffman bound)**.** *Let G be a graph on n vertices which is regular of degree d and let λ_{min} be the least eigenvalue of G. Then for any independent set S in G*

$$|S| \leqslant \frac{-n\lambda_{min}}{d - \lambda_{min}}$$

Note that the right hand side of the inequality is actually positive, since the least eigenvalue of a simple graph is negative (recall that $\mathrm{Tr}G = 0$).

Proof. Let A be the adjacency matrix of G and let $\lambda = \lambda_{min}$ be its least eigenvalue. First notice that since G is a regular graph, the sum of any row of A is equal to d, so $\mathbf{j}$ is an eigenvector of A with eigenvalue d. We then have

$$(A - \lambda I)\mathbf{j} = (d - \lambda)\mathbf{j}$$

and if we define

$$M := A - \lambda I - \frac{d - \lambda}{n}J$$

we have $M\mathbf{j} = 0$ since $J\mathbf{j} = n\mathbf{j}$ and $A\mathbf{j} = d\mathbf{j}$. Now let $\mathbf{u}$ be any other eigenvector of A orthogonal to $\mathbf{j}$ so $J\mathbf{u} = 0$. This implies that $\mathbf{u}$ is also an eigenvector of M. In addition, an easy computation shows that this eigenvector corresponds to a non-negative eigenvalue, so matrix M is positive semidefinite. Now let S be an independent set of G with $|S| = m$, and let $\phi = (\phi_1, \cdots, \phi_n)^\mathsf{T}$ where $\phi_i = 1$ if $i \in S$, $\phi_i = 0$ otherwise. Clearly we have $\phi^\mathsf{T} A\phi = 0$, $\phi^\mathsf{T}\phi = m$ and also $\phi^\mathsf{T} J\phi = m^2$, so:

$$0 \leqslant \phi^\mathsf{T} M\phi = -\lambda\phi^\mathsf{T}\phi - \frac{d - \lambda}{n}\phi^\mathsf{T} J\phi = -\lambda m - \frac{d - \lambda}{n}m^2$$

and solving for m we get the desired result. ■

Corollary 2.9. *The independency number of a d-regular graph G on n vertices satisfies:*

$$\alpha(G) \leqslant \frac{-n\lambda_{min}}{d - \lambda_{min}}$$

Proof. From the definition of $\alpha(G)$ we have that $\alpha(G) = |S|$ for some independent set S of G. ∎

Proposition 2.10. *Let G be a d-regular graph on n vertices and suppose that $d \neq \lambda \in spec(G)$. Then $-1-\lambda \in spec(\overline{G})$. Moreover, $n-1-d \in spec(\overline{G})$.*

Proof. Let A be the adjacency matrix of G, so that the matrix $J - I - A$ is the adjacency matrix of the complement $\overline{G}$. The vector $\mathbf{j}$ satisfies $A\mathbf{j} = d\mathbf{j}$ and thus $(J-I-A)\mathbf{j} = n-1-d$, so $n-1-d \in \text{spec}(\overline{G})$. Now let $\lambda \neq d$ be an eigenvalue of A and let $\mathbf{v}$ be its corresponding eigenvector. Then:

$$(J-I-A)\mathbf{v} = (-1-\lambda)\mathbf{v}$$

and therefore $-1-\lambda \in \text{spec}(\overline{G})$. ∎

Corollary 2.11. *If λ_{max} is the largest eigenvalue of the spectrum of a d-regular graph on n vertices, then*

$$\omega(G) \leqslant \frac{n(1+\lambda_{max})}{n-d+\lambda_{max}}$$

Proof. $\overline{G}$ is a regular graph of degree $n-1-d$. Let λ'_{min} be the least eigenvalue of $\overline{G}$. Then by proposition 2.10 we have that $\lambda'_{min} = -1-\lambda_{max}$, where λ_{max} is the greatest eigenvalue of G. Applying Theorem 2.8 to $\overline{G}$:

$$\omega(G) = \alpha(\overline{G}) \leqslant \frac{-n\lambda'_{min}}{n-1-d-\lambda'_{min}} = \frac{n(1+\lambda_{max})}{n-d+\lambda_{max}}.$$

∎

Note that the bound given by this last corollary is optimal, for instance is met by all K_n. In that case we have $d = n-1$ and $\lambda_{max} = n-1$ so

$$n = \omega(K_n) \leqslant \frac{n(1+(n-1))}{n-(n-1)+(n-1)} = n.$$

2.4 Spectra of strong products

We shall now derive the spectrum of a strong product of graphs in terms of the spectrum of each one. We start by recalling some definitions on linear algebra regarding products.

Definition 2.12 (Inner product)**.** Let $\mathbf{u}, \mathbf{v}$ be two vectors of the ususal euclidean space $\mathbf{R}^n$. The *inner product* of $\mathbf{u}$ and $\mathbf{v}$ is:

$$\mathbf{u}^\mathsf{T}\mathbf{v} = u_1v_1 + \cdots + u_nv_n$$

Definition 2.13 (Tensor product of vectors)**.** Let $\mathbf{u} \in \mathbf{R}^n$ and $\mathbf{v} \in \mathbf{R}^m$. We define the *tensor product* of $\mathbf{u}$ and $\mathbf{v}$ to be:

$$\mathbf{u} \otimes \mathbf{v} = (u_1v_1, \dots, u_1v_m, u_2v_1, \dots, \dots, u_nv_m)^\mathsf{T} \in \mathbf{R}^{nm}$$

An easy computation shows that these two definitions are related in the following way. If we let $\mathbf{x}, \mathbf{v} \in \mathbf{R}^n$ and $\mathbf{y}, \mathbf{w} \in \mathbf{R}^m$, then the following holds:

$$(\mathbf{x} \otimes \mathbf{y})^\mathsf{T}(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{x}^\mathsf{T}\mathbf{u})(\mathbf{y}^\mathsf{T}\mathbf{v}). \tag{2.2}$$

Definition 2.14 (Tensor product of matrices)**.** Let A be a $m \times n$ matrix and let B be a $p \times q$ matrix. Then the *tensor product* (also called *Kronecker product*) of A and B is the $mp \times nq$ matrix given by

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

where $A = (a_{ij})$ are the entries of A.

Again a routinary computation shows that if A is a $m \times n$ matrix and B is a $p \times q$ matrix, then for $\mathbf{u}$ and $\mathbf{v}$ vectors of $\mathbf{R}^n$ and $\mathbf{R}^q$ respectively, we have:

$$(A \otimes B)(\mathbf{u} \otimes \mathbf{v}) = (A\mathbf{u}) \otimes (B\mathbf{v}).$$

Theorem 2.15. *Let G and H be two graphs with spectra $spec(G) = \{\lambda_1, \dots, \lambda_n\}$ and $spec(H) = \{\mu_1, \dots, \mu_m\}$. We then have:*

$$spec(G \boxtimes H) \ = \big\{(\lambda_i + 1)(\mu_j + 1) - 1 \ : \ 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant m\big\}.$$

Proof. Let A and B be the adjacency matrices of G and H respectively. We shall first show that the adjacency matrix of $G \boxtimes H$ is the matrix $(A+I) \otimes (B+I) - I$. For that, let (x_1, y_1) and (x_2, y_2) be two vertices of $G \boxtimes H$. We compute

$$\begin{aligned}
&(\mathbf{e}_{x_1} \otimes \mathbf{e}_{y_1})^\mathsf{T}\Big((A+I) \otimes (B+I) - I\Big)(\mathbf{e}_{x_2} \otimes \mathbf{e}_{y_2}) \\
&= \quad (\mathbf{e}_{x_1} \otimes \mathbf{e}_{y_1})^\mathsf{T}\Big((A+I) \otimes (B+I)\Big)(\mathbf{e}_{x_2} \otimes \mathbf{e}_{y_2}) - (\mathbf{e}_{x_1} \otimes \mathbf{e}_{y_1})^\mathsf{T}(\mathbf{e}_{x_2} \otimes \mathbf{e}_{y_2}) \\
&= \quad \Big(\mathbf{e}_{x_1}^\mathsf{T}(A+I)\mathbf{e}_{x_2}\Big)\Big(\mathbf{e}_{y_1}^\mathsf{T}(B+I)\mathbf{e}_{y_2}\Big) - (\mathbf{e}_{x_1}^\mathsf{T}\mathbf{e}_{x_2})(\mathbf{e}_{y_1}^\mathsf{T}\mathbf{e}_{y_2}) \\
&= \quad (a_{x_1,x_2} + \delta_{x_1,x_2})(b_{y_1,y_2} + \delta_{y_1,y_2}) - \delta_{x_1,x_2}\delta_{y_1,y_2}
\end{aligned}$$

which is equal to 1 if and only if vertices (x_1, y_1) and (x_2, y_2) are adjacent.

Now let $\mathbf{u}$ and $\mathbf{v}$ be two eigenvectors of A and B with corresponding eigenvalues λ and μ, respectively. In this case:

$$\begin{aligned}
\Big((A+I) \otimes (B+I) - I\Big)(\mathbf{u} \otimes \mathbf{v}) &= \Big((A+I) \otimes (B+I)\Big)(\mathbf{u} \otimes \mathbf{v}) - (\mathbf{u} \otimes \mathbf{v}) \\
&= \Big((\lambda+1)\mathbf{u}\Big) \otimes ((\mu+1)\mathbf{v}\Big) - (\mathbf{u} \otimes \mathbf{v}) \\
&= \Big((\lambda+1)(\mu+1) - 1\Big)(\mathbf{u} \otimes \mathbf{v})
\end{aligned}$$

and therefore $\mathbf{u} \otimes \mathbf{v}$ is an eigenvector of $G \boxtimes H$ with eigenvalue $(\lambda+1)(\mu+1) - 1$. ∎

3. Lovász number

In this chapter we will present a graph parameter introduced by Lovász on [2] called *Lovász number*, that actually serves as an upper bound on the Shannon capacity of that graph. We will then compute the value of this number for the pentagon, which will allow us to determine its Shannon capacity. Then we will show that the Hoffman bound given in last section is also satisfied by the Lovász number, and we will give a characterization of the number in question in terms of the spectrum of a set of matrices. We will end up by determining the Lovász number for a particular set of symmetric graphs and apply this result to the cycle graphs. Finally we will be notting the dificulties that present the odd cycles, such as the Heptagon.

3.1 Orthonormal representations and Lovász ϑ function

Definition 3.1 (Orthonormal representation)**.** Let G be a graph on n vertices. An *orthonormal representation* (or *orthogonal labeling*) of G is an assignment of vectors $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ of some Euclidean space $\mathbf{R}^d$ to each of the n vertices of G, such that if two vertices i and j are not adjacent then their corresponding vectors $\mathbf{u}_i$ and $\mathbf{u_j}$ are orthogonal.

Note that dimension of the space where the vectors belong is not specified, so any graph G on n vertices has a trivial orthonormal representation given by any orthonormal basis of $\mathbf{R}^n$.

Lemma 3.2. *Let $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ and $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ be two orthonormal representations of graphs G and H respectively.Then $(\mathbf{u}_1 \otimes \mathbf{v}_1, \ldots, \mathbf{u}_n \otimes \mathbf{v}_m)$ is an orthonormal representation of $G \boxtimes H$.*

Proof. Let (x_1, x_2) and (y_1, y_2) be two vertices of $G \boxtimes H$ which are not adjacent, so one of $\mathbf{u}_{x_1}^\mathsf{T} \mathbf{v}_{y_1} = 0$ or $\mathbf{u}_{x_2}^\mathsf{T} \mathbf{v}_{y_2}$ is 0. Using relation (2.2):

$$(\mathbf{u}_{x_1} \otimes \mathbf{u}_{x_2})^\mathsf{T} (\mathbf{v}_{y_1} \otimes \mathbf{v}_{y_2}) = (\mathbf{u}_{x_1}^\mathsf{T} \mathbf{v}_{y_1})(\mathbf{u}_{x_2}^\mathsf{T} \mathbf{v}_{y_2}) = 0$$

so vectors $\mathbf{u}_{x_1} \otimes \mathbf{u}_{x_2}$ and $\mathbf{v}_{y_1} \otimes \mathbf{v}_{y_2}$ are orthogonal. ■

We define the *value* of an orthonormal representation to be

$$\min_{\mathbf{c}} \max_{1 \leqslant i \leqslant n} \frac{1}{(\mathbf{c}^\mathsf{T} \mathbf{u}_i)^2}$$

where $\mathbf{c}$ ranges over all unit vectors. The vector $\mathbf{c}$ yielding such a minimum is called the *handle* of the representation $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$. Now if we seek the minimum value attainable by ranging over all the orthonormal representation of a given graph G, we end up with the graph parameter called the Lovász number of G. More formally:

Definition 3.3 (Lovász number)**.** Let G be a graph on n vertices. The *Lovász number* $\vartheta(G)$ of G is the minimum value over all the orthonormal representations of G. That is:

$$\vartheta(G) = \min_{(\mathbf{u}_1, \ldots, \mathbf{u}_n)} \min_{\mathbf{c}} \max_{1 \leqslant i \leqslant n} \frac{1}{(\mathbf{c}^\mathsf{T} \mathbf{u}_i)^2}.$$

A representation which attains this minimum is called an *optimal representation*. As we are going to see next, the number $\vartheta(G)$ of a graph G is closely related to the Shannon capacity of G. We need first two lemmas:

Lemma 3.4. $\vartheta(G \boxtimes H) \leqslant \vartheta(G)\vartheta(H)$.

Proof. Let $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ and $(\mathbf{v}_1, \dots, \mathbf{v}_m)$ be two optimal orthonormal representations for G and H with handles $\mathbf{c}$ and $\mathbf{d}$ respectively. On the one hand $(\mathbf{u}_1 \otimes \mathbf{v}_1, \dots, \mathbf{u}_n \otimes \mathbf{v}_m)$ is an orthonormal representation (not necessarily optimal) of $\vartheta(G \boxtimes H)$ by lemma 3.2. On the other, the vector $\mathbf{c} \otimes \mathbf{d}$ is unitary by relation (2.2), so we have:

$$\vartheta(G \boxtimes H) \leqslant \min_{\mathbf{e}} \max_{i,j} \frac{1}{\left(\mathbf{e}^{\mathsf{T}}\left(\mathbf{u}_i \otimes \mathbf{v}_j\right)\right)^2} \leqslant \max_{i,j} \frac{1}{\left((\mathbf{c} \otimes \mathbf{d})^{\mathsf{T}}(\mathbf{u}_i \otimes \mathbf{v}_j)\right)^2}$$
$$\leqslant \max_{i,j} \frac{1}{(\mathbf{c}^{\mathsf{T}}\mathbf{u}_i)^2} \cdot \frac{1}{(\mathbf{d}^{\mathsf{T}}\mathbf{v}_j)^2} = \vartheta(G)\vartheta(H).$$

■

Lemma 3.5. $\alpha(G) \leqslant \vartheta(G)$

Proof. Without loss of generality we let $\{1, \dots, k\}$ be a maximum independent set of vertices of G and $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ be an optimal orthonormal representation of G with handle $\mathbf{c}$. Then $\mathbf{u}_1, \dots, \mathbf{u}_k$ are pairwise orthogonal and therefore

$$1 = \frac{1}{\mathbf{c}^{\mathsf{T}}\mathbf{c}} \leqslant \frac{1}{\sum_{i=1}^{k}(\mathbf{c}^{\mathsf{T}}\mathbf{u}_i)^2} \leqslant \frac{1}{k \cdot \min\limits_{1 \leqslant i \leqslant k} (\mathbf{c}^{\mathsf{T}}\mathbf{u}_i)^2} = \frac{1}{k} \cdot \max_{1 \leqslant i \leqslant k} \frac{1}{(\mathbf{c}^{\mathsf{T}}\mathbf{u}_i)^2} \leqslant \frac{\vartheta(G)}{k}$$

and since $k = \alpha(G)$ the claim follows. ■

Theorem 3.6. *The Shannon capacity of a graph G is bounded by its Lovász number. That is*

$$\Theta(G) \leqslant \vartheta(G).$$

Proof. From Lemmas 3.5 and 3.4 respectively we have:

$$\alpha(G^n) \leqslant \vartheta(G^n) \leqslant \vartheta(G)^n,$$

so

$$\Theta(G) = \sup_n \alpha(G^n)^{1/n} \leqslant \vartheta(G).$$

■

3.2 Capacity of the Pentagon

With all the machinery we have developed we are ready to compute the actual value $\Theta(C_5)$ which we left pending on the first section. We state it in the form of the following theorem.

Theorem 3.7 (Capacity of C_5)**.** *Let C_5 be the cycle graph on 5 vertices. Then*

$$\Theta(C_5) = \sqrt{5}.$$

Proof. We already know $\sqrt{5} \leqslant \Theta(C_5)$. Rigorously, take $S = \{13, 21, 34, 42, 55\}$ as an independent subset of vertices in $C_5 \boxtimes C_5$. This shows $5 \leqslant \alpha(C_5^2)$ and therefore:

$$\sqrt{5} \leqslant \sqrt{\alpha(C_5^2)} \leqslant \sup_n \alpha(C_5^n)^{\frac{1}{n}} = \Theta(C_5).$$

To prove the other inequality, we are going to compute $\vartheta(C_5)$. Consider an umbrella consisting of a handle of unit length and five ribs holding the cloth, each one of unit length too (figure 9). Assume the top of the umbrella is at the origin O of the 3-dimensional Euclidean space $\mathbf{R}^3$, and the end of the handle is at the point $(0, 0, 1)$. When the umbrella is closed, the ends of the five ribs, say $A_1, \dots, A_5$, are also at the point $(0, 0, 1)$. As the umbrella opens (uniformly), these ends form a regular pentagon lying on a circle with center $(0, 0, \sqrt{1 - r^2})$ and radius r, parallel to the plane XY. We are going to find the particular value of r such that any nonconsecutive pair of ribs' endpoints A_i and A_j give rise to orthogonal vectors $\vec{OA_i}$ and $\vec{OA_j}$.

In a regular pentagon inscribed in a circle of radius r, the length of each side of the pentagon is equal to $2r\sin(\pi/5)$, and the distance between non-adjacent vertices is equal to $2r\sin(2\pi/5)$. If we let A_i and A_j be such a pair of vertices, the isosceles triangle OA_iA_j has two legs of length 1 and an hypotenuse of length $2r\sin(2\pi/5)$. By imposing this triangle to be rectangle (so that sides OA_i and OA_j form a right angle) we must have

$$1^2 + 1^2 = (2r\sin(2\pi/5))^2 \quad \textit{(Pythagoras)}$$

which gives

$$r = \frac{\csc(2\pi/5)}{\sqrt{2}} < 1$$

and thus for such a value of r, two nonconsecutive A_i and A_j give place to orthogonal vectors $\vec{OA_i}$ and $\vec{OA_j}$. By defining $\mathbf{u}_i = \vec{OA_i}$, we get a set $(\mathbf{u}_1, \dots, \mathbf{u}_5)$ which is an orthonormal representation of C_5. In addition, if we take the handle of the representation $\mathbf{c}$ to be equal to the *actual* handle of the umbrella and using the fact that $\cos(2\pi/5) = \frac{1}{4}(-1 + \sqrt{5})$ we have

$$\vartheta(G) \leqslant \frac{1}{(\mathbf{c}^\mathsf{T}\mathbf{u}_i)^2} = \frac{1}{1 - r^2} = \sqrt{5}$$

and so by Theorem 3.6 we get $\Theta(C_5) \leqslant \sqrt{5}$, from which it follows finally that $\Theta(C_5) = \sqrt{5}$. ∎

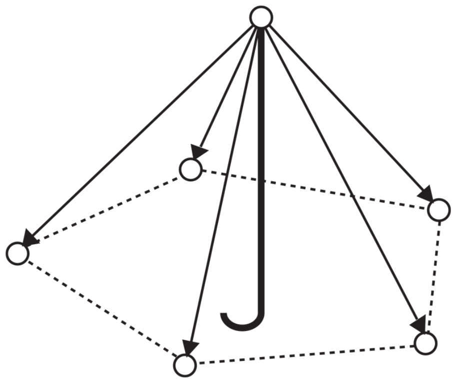


Figure 9: Lovász umbrella

3.3 Spectral bound of the Lovász number

The proof of Theorem 3.7 implied that $\vartheta(C_5) \leqslant \sqrt{5}$. This bound is the same as the bound provided by Theorem 2.8 on the independency number of C_5, and as we will see next this is not a coincidence.

Theorem 3.8. *Let G be a graph on n vertices which is regular of degree d and let λ_{min} be the least eigenvalue of G. Then*

$$\vartheta(G) \leqslant \frac{-n\lambda_{min}}{d - \lambda_{min}} \tag{3.1}$$

Proof. Let A be the adjacency matrix of G and let $\lambda = \lambda_{min}$. In the proof of Theorem 2.8 we already saw that the matrix $M = A - \lambda I - \frac{d-\lambda}{n} J$ was positive semidefinite, so by proposition 2.4 we can write $M = BB^{\mathsf{T}}$ for some real matrix B. If we let the rows of B be $\mathbf{u}_1, \dots, \mathbf{u}_n$ we have:

$$\begin{aligned}
\mathbf{u}_i^{\mathsf{T}}\mathbf{u}_j = \mathbf{e}_i^{\mathsf{T}} M \mathbf{e}_j &= \mathbf{e}_i^{\mathsf{T}} \left(A - \lambda I - \frac{d-\lambda}{n} \right) \mathbf{e}_j \\
&= \mathbf{e}_i^{\mathsf{T}} A \mathbf{e}_j - \lambda \mathbf{e}_i^{\mathsf{T}} \mathbf{e}_j - \frac{d-\lambda}{n} \mathbf{e}_i J \mathbf{e}_j \\
&= a_{ij} - \delta_{ij} - \frac{d-\lambda}{n}
\end{aligned}$$

where $\mathbf{e}_i$ is the i-th canonical basis vector. Now notice that since M is singular (we saw already that $M\mathbf{j} = 0$) then B does not have full rank and therefore there is a vector $\mathbf{c}$ orthogonal to each of its rows, i.e. $\mathbf{c}^{\mathsf{T}}\mathbf{u}_i = 0$ for all i. Taking $\mathbf{c}$ to be unitary and defining

$$\mathbf{v}_i := \frac{1}{\sqrt{-\lambda}} \mathbf{u}_i - \frac{1}{\sqrt{-\lambda n/(d-\lambda)}} \mathbf{c}$$

we have that for nonadjacent vertices i, j (that is, $a_{ij} = 0$), the product

$$\begin{aligned}
\mathbf{v}_i^{\mathsf{T}}\mathbf{v}_j &= \left(\frac{1}{\sqrt{-\lambda}} \right)^2 \mathbf{u}_i^{\mathsf{T}}\mathbf{u}_j + \left(-\frac{1}{\sqrt{-\lambda n/(d-\lambda)}} \right)^2 \mathbf{c}^{\mathsf{T}}\mathbf{c} \\
&= \left(\frac{1}{\sqrt{-\lambda}} \right)^2 \cdot \left(-\frac{d-\lambda}{n} \right) + \left(-\frac{1}{\sqrt{-\lambda n/(d-\lambda)}} \right)^2 = 0.
\end{aligned}$$

so $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ is an orthonormal representation of G. Finally, for any i

$$\frac{1}{(\mathbf{c}^\mathsf{T}\mathbf{v}_i)^2} = \frac{-\lambda n}{d - \lambda}$$

and by the definition of $\vartheta(G)$ the result follows. ■

In the case when G is the cyclic graph C_5, we have that G is regular of degree 2. In addition $\text{spec}(G) = \{2, \frac{1}{2}(-1+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5})\}$ (the last two with multiplicity 2), so the least eigenvalue of G is $\lambda_{min} = \frac{1}{2}(-1-\sqrt{5})$. By compputing the bound on (3.1)

$$\frac{-n\lambda_{min}}{d - \lambda_{min}} = \frac{-5 \cdot \frac{1}{2}(-1-\sqrt{5})}{2 - \frac{1}{2}(-1-\sqrt{5})} = \sqrt{5}$$

so Theorem 3.8 provides an alternative proof of $\Theta(C_5) = \sqrt{5}$.

The following Theorem gives us an alternative definition for $\vartheta(G)$ related to the spectrum of matrices.

Theorem 3.9. *Let G be a graph on vertices $\{1, \dots, n\}$. Then $\vartheta(G)$ is the minimum attainable for the largest eigenvalue of any symmetric matrix $A = (a_{ij})$ such that*

$$a_{ij} = 1 \quad \text{if } i = j \text{ or if } i \text{ and } j \text{ are non-adjacent.} \tag{3.2}$$

Proof. First of all, let A be any matrix satisfying equation (3.2) and let λ be its largest eigenvalue and let $\mathbf{c}$ be its corresponding unitary eigenvector. Then $\lambda I - A$ is positive semidefinite, so by proposition 2.4 we have $\lambda I - A = V^\mathsf{T} V$ for some matrix V. If we let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the rows of V, we have

$$\begin{aligned} \mathbf{v}_i^\mathsf{T}\mathbf{v}_j &= -1 \qquad \text{if } i \neq j \text{ and } i,j \text{ are non-adjacent} \\ \mathbf{v}_i^\mathsf{T}\mathbf{v}_i &= \lambda - 1 \end{aligned}$$

and also $\mathbf{c}^\mathsf{T}\mathbf{v}_i = 0$. Now define

$$\mathbf{u}_i = \frac{1}{\sqrt{\lambda}}(\mathbf{c} + \mathbf{v}_i)$$

so that for distinct and non-adjacent i, j we have:

$$\mathbf{u}_i^\mathsf{T}\mathbf{u}_j = \frac{1}{\lambda}(\mathbf{c} + \mathbf{v}_i)^\mathsf{T}(\mathbf{c} + \mathbf{v}_j) = \frac{1}{\lambda}(\mathbf{c}^\mathsf{T}\mathbf{c} + \mathbf{c}^\mathsf{T}\mathbf{v}_j + \mathbf{v}_i^\mathsf{T}\mathbf{c} + \mathbf{v}_i^\mathsf{T}\mathbf{v}_j) = \frac{1}{\lambda}(1 + 0 + 0 - 1) = 0,$$

and also

$$\mathbf{u}_i^\mathsf{T}\mathbf{u}_i = \frac{1}{\lambda}(\mathbf{c} + \mathbf{v}_i)^\mathsf{T}(\mathbf{c} + \mathbf{v}_i) = \frac{1}{\lambda}(\mathbf{c}^\mathsf{T}\mathbf{c} + 2 \cdot \mathbf{c}^\mathsf{T}\mathbf{v}_i + \mathbf{v}_i^\mathsf{T}\mathbf{v}_i) = \frac{1}{\lambda}(1 + 0 + \lambda - 1) = 1.$$

Therefore $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ is an orthonormal representation of G which satisfies

$$\frac{1}{(\mathbf{c}^\mathsf{T}\mathbf{u}_i)^2} = \frac{1}{\left(\frac{1}{\sqrt{\lambda}} \cdot \mathbf{c}^\mathsf{T}(\mathbf{c} + \mathbf{v}_i)\right)^2} = \lambda$$

and from the definition of $\vartheta(G)$ we conclude that $\vartheta(G) \leqslant \lambda$.

We now will show that this minimum can actually be attained. Let $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ be a optimal orthonormal representation of G with handle $\mathbf{c}$ and let A be the matrix whose entries are:

$$\begin{aligned} a_{ij} &= 1 - \frac{\mathbf{u}_i^\mathsf{T}\mathbf{u}_j}{(\mathbf{c}^\mathsf{T}\mathbf{u}_i)(\mathbf{c}^\mathsf{T}\mathbf{u}_j)}, \quad i \neq j, \\ a_{ii} &= 1 \end{aligned}$$

Clearly A satisfies condition (3.2). Moreover their entries also satisfy:

$$\begin{aligned} -a_{ij} &= \left(\mathbf{c} - \frac{\mathbf{u}_i}{\mathbf{c}^\mathsf{T}\mathbf{u}_i}\right)^\mathsf{T}\left(\mathbf{c} - \frac{\mathbf{u}_j}{\mathbf{c}^\mathsf{T}\mathbf{u}_j}\right) \\ \vartheta(G) - a_{ii} &= \left(\mathbf{c} - \frac{\mathbf{u}_i}{\mathbf{c}^\mathsf{T}\mathbf{u}_i}\right)^2 + \left(\vartheta(G) - \frac{1}{(\mathbf{c}^\mathsf{T}\mathbf{u}_i)^2}\right) \end{aligned}$$

so if we define $\mathbf{v}_i = \mathbf{c} - \frac{\mathbf{u}_i}{\mathbf{c}^\mathsf{T}\mathbf{u}_i}$, $V = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ and $D = \operatorname{diag}\left(\vartheta(G) - \frac{1}{(\mathbf{c}^\mathsf{T}\mathbf{u}_1)^2}, \dots, \vartheta(G) - \frac{1}{(\mathbf{c}^\mathsf{T}\mathbf{u}_n)^2}\right)$ we have that

$$\vartheta(G)I - A = V^\mathsf{T}V + D.$$

Now D is a diagonal matrix with positive entries (recall $\vartheta(G) \geqslant \frac{1}{(\mathbf{c}^T\mathbf{u}_i)^2}$ $\forall i$ since $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ is optimal with handle $\mathbf{c}$), and thus positive semidefinite. On the other hand $V^\mathsf{T}V$ is positive semidefinite too by proposition 2.4, so we conclude that $\vartheta(G)I - A$ is positive semidefinite, implying that the largest eigenvalue of A can be at most $\vartheta(G)$. ■

3.4 Lovász number of Odd Cycles

The fact that we have been able to compute Shannon's capacity for C_5 is due to the *accidentallity* of $\alpha(C_5^k)^{\frac{1}{k}}$ being equal to $\vartheta(C_5)$ for $k = 2$, which caused the number $\Theta(C_5)$ to get absolutely squeezed between them. The same thing happens again if we consider C_3 since:

$$1 = \alpha(C_3) \leqslant \Theta(C_3) \leqslant \vartheta(C_3) \leqslant \frac{-3(-1)}{2-(-1)} = 1$$

Also for the even cycles C_{2n} :

$$n = \alpha(C_{2n}) \leqslant \Theta(C_{2n}) \leqslant \vartheta(C_{2n}) \leqslant \frac{(-2n)\cdot(-2)}{2-(-2)} = n$$

using in both cases Theorem 3.8. However, if we try the same approach with odd cycle graphs other than 3 or 5 (such as C_7) we won't get that lucky, as we are going to see down below.

The goal here will be to compute the actual value of $\vartheta(G)$ for all odd cycle graphs by exploiting the fact that those are somewhat *symmetric*. Lets first define rigorously what we mean by a graph being symmetric:

Definition 3.10 (Graph automorphism)**.** Let G be a graph on vertices $V(G)$. A permutation σ acting on the vertex set of G is called a *graph automorphism* if for any pair (x, y) of vertices of G we have that x, y are adjacent if and only if $\sigma(x), \sigma(y)$ are adjacent. The group defined by $\Gamma(G) = \{\sigma : \sigma \text{ is an automorphism of } G\}$ is called the *automorphism group of G*.

Definition 3.11 (Transitivity)**.** A graph G is a *vertex transitive* graph if for any pair of vertices x, y there is an automorphism of G sending x to y. Similarly, a graph G is an *edge transitive* graph if for any two edges e, f there is an automorphism of G sending e to f.

Some examples of automorphism groups for simple graphs are $\Gamma(K_n) = S_n$ and $\Gamma(C_n) = D_n$, the symmetric group on n elements and the dihedral group on the n-gon respectively. In both cases, those groups act transitively on the vertices and on the edges. A good property of edge transitive graphs is that when they are regular too, inequality provided by Theorem 3.8 becomes a very useful equality. We shall state this in the form of the following Theorem:

Theorem 3.12. *Let G be an edge-transitive graph on n vertices which is regular of degree d and let λ_{min} be the least eigenvalue of G. Then*

$$\vartheta(G) = \frac{-n\lambda_{min}}{d - \lambda_{min}} \tag{3.3}$$

Proof. Let $\lambda = \lambda_{min}$ and let G have vertices $\{1, \dots, n\}$. From Theorem 3.8 we already have $\vartheta(G) \leqslant (-n\lambda)/(d - \lambda)$. For the opposite inequality, let $C = (c_{ij})$ be a symmetric matrix such that $c_{ij} = 1$ if i, j are equal or adjacent in G, with its largest eigenvalue equal to $\vartheta(G)$. Now let $\Gamma(G)$ be the group of automorphisms of G and consider

$$C' = \frac{1}{|\Gamma(G)|} \sum_{P \in \Gamma(G)} P^{-1}CP$$

where the elements of $\Gamma(G)$ are taken as $n \times n$ permutation matrices. Since G is edge-transitive, C' is then of the following form:

$$c'_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or if } i, j \text{ are non-adjacent} \\ \beta & \text{otherwise} \end{cases}$$

and its largest eigenvalue is at most $\vartheta(G)$. Since C' also satisfies the conditions of Theorem 3.9, we conclude that is in fact equal to $\vartheta(G)$.

On the other hand, as in the proof of Theorem 3.8, if $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ are the eigenvalues of the adjacency matrix A of G, then the eigenvalues of any matrix $J - xA$ are $n - x\lambda_1, -x\lambda_2, \dots, -x\lambda_n$ with $\lambda_1 = d$. The greatest of them is either the first or the last, and the optimal case is given when they are both equal. Since matrix C' has its largest eigenvalue equal to $\vartheta(G)$ and is of the form $J - xA$ the claim follows. ∎

Corollary 3.13. *For odd n,*

$$\vartheta(C_n) = \frac{n\cos(\pi/n)}{1 + cos(\pi/n)}$$

Proof. C_n is a 2-regular edge-transitive graph with least eigenvalue $\lambda_{min} = -2\cos(\pi/n)$. Applying Theorem 3.12 the claim follows. ∎

Lets try now to generalize the aproach of squeezing the value of $\Theta(C_5)$ for all odd circuits. More specific, we are looking for some k for which the independency number of C_n^k equals $\vartheta(C_n)^k$, since then we would have

$$\Theta(C_n) \geqslant \alpha(C_n^k)^{\frac{1}{k}} = \vartheta(C_n) \geqslant \Theta(C_n)$$

implying $\Theta(C_n) = \vartheta(C_n)$. However, note that while $\alpha(G^k)$ is always an integer (it is the cardinality of some subset of the vertex set of G^k), equation (3.3) shows that no power of $\vartheta(C_n)$ is an integer for odd $n > 5$. This implies that if it where the case that $\Theta(C_n) = \vartheta(C_n)$ for n odd, then $\sup_k \alpha(C_n^k)$ is never attained for finite k. The value of $\Theta(C_7)$ still remains unknown...

4. Kneser graphs

Kneser graphs where introduced by Kneser on [4] and are relevant to this thesis as they serve as an other example of a non trivial family of graphs whose Shannon capacity can be determined using the spectral characterization of the Lovász number. They also include a notable studied family of symmetric graphs named *odd graphs*.

In this section we will define what are Kneser graphs and remark some of its properties. We will then compute the spectrum of this family of graphs and use it again as a tool to derive other parameters of the family, such as their independency number and their Shannon Capacity as said before. We will end up by commenting a result due to Lovász regarding the chromatic number of the family.

Definition 4.1 (Kneser graph)**.** Let n and k be positive integers with $2k \leqslant n$. The *Kneser Graph* $KG_{n,k}$ is the graph whose vertices are the k-element subsets of $\{1, \dots, n\}$ and where two vertices are adjacent if and only if their corresponding sets are disjoint.

Note that if $n/2 < k < n$, then the intersection of any two k subsets of $\{1, \dots, n\}$ is empty, so $KG_{n,k}$ has no edges. Moreover, if $k = n$ the graph only has one vertex and if $k > n$ then the vertex set is empty.

For $k = 1$ the Kneser graph $KG_{n,1}$ is trivially the complete graph K_n. The first non-trivial kneser graph is the notorious Petersen Graph, which corresponds to $K_{5,2}$.

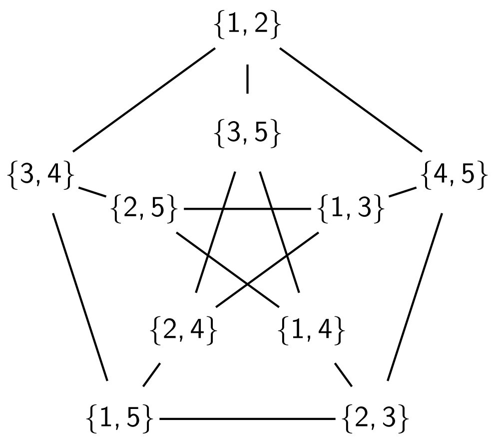


Figure 10: Petersen graph shown as the non-intersecting 2-subsets of $\{1, 2, 3, 4, 5\}$

The spectrum of all Kneser graphs is characterized by the following proposition:

Proposition 4.2. *The eigenvalues of the Kneser graph* $K_{n,k}$ *are*

$$(-1)^t \binom{n-k-t}{k-t} \quad t = 0, 1, \dots, k$$

Proof. First of all note that $KG_{n,k}$ is a regular graph of degree $\binom{n-k}{k}$ since any k-subset S of $\{1, \dots, n\}$ is disjoint to any other k-subset of $\{1, \dots, n\} \backslash S$, and the number of such subsets is $\binom{n-k}{k}$. Therefore $\mathbf{j}$ is an eigenvector of eigenvalue $\binom{n-k}{k}$. In order to find the remaining eigenvalues we will look for plausible eigenvectors first.

Suppose that $\mathbf{x} = (x_K)$ is an eigenvector of the adjacency matrix of the Kneser graph $K_{n,k}$, where the coordinates of $\mathbf{x}$ are indexed by k-subsets $K \subset \{1, \dots, n\}$. Then, for each of those subsets K we must have:

$$\sum_{K' \cap K = \emptyset} x_{K'} = \lambda x_K \tag{a}$$

with λ the corresponding eigenvalue.

One way to obtain x^K for every K satisfying (a) for some λ and all K is as follows. Fix $t < k$ and consider numbers $\{y^T\}$ to be specified later where $T \subset \{1, \dots, n\}$ and $|T| = t$. For each k-subset K define

$$x_K = \sum_{\substack{T \subset K \\ |T| = t}} y_T.$$

We observe that every t-subset T not meeting K appears in $\binom{n-k-t}{k-t}$ k-subsets disjoint from K: once T is fixed we must complete it with $k - t$ elements not in $K \cup T$ to obtain a k-subset K' disjoint from K and containing T. Therefore equation (a) reads

$$\sum_{K' \cap K = \emptyset} x_{K'} = \sum_{T \cap K = \emptyset} \binom{n-k-t}{k-t} y_T = \lambda \sum_{T \subset K} y_T \tag{b}$$

We would be done if we could find real numbers y_T such that, for each k-subset K of $\{1, \dots, n\}$, we had

$$\sum_{T \cap K} y_T = (-1)^t \sum_{T \subset K} y_T \tag{c}$$

since then (b) would give the eigenvalue

$$\lambda = (-1)^t \binom{n-k-t}{k-t}.$$

Let us show that there are in fact $\binom{n}{t} - \binom{n}{t-1}$ independent vectors $\mathbf{y} = (y_T)$ satisfying equation (c). Actually it suffices to show that for each $U \subset \{1, \dots, n\}$ with $|U| = t - 1$ one has

$$\sum_{T \supset U} y_T = 0. \tag{d}$$

If this is the case then, for each $i = 0, 1, \dots, t$ one has:

$$0 = \sum_{\substack{U \\ |U \cap K| = i}} \sum_{\substack{T \\ T \supset U}} y_T = \sum_{\substack{T \\ |T \cap K| = i}} \sum_{\substack{U \\ U \subset T}} y_T + \sum_{\substack{T \\ |T \cap K| = i+1}} \sum_{\substack{U \\ U \subset T}} y_T \tag{e}$$

because, whenever $|U \cap K| = i$, sets T containing U satisfy $i \leqslant |T \cap K| \leqslant i + 1$, and the second inequality in (e) follows by exchanging the order of summation. Now, for every set T with $|T \cap K| = i + 1$ we have precisely $(t - i)$ sets U contained in T with $|U \cap K| = i$, while if $|T \cap K| = i$ then T contains $i + 1$ sets U contained in T with $|U \cap K| = i$. Therefore equation (e) gives

$$0 = (t - i) \left(\sum_{\substack{T \\ |T \cap K| = i}} y_T \right) + (i + 1) \left(\sum_{\substack{T \\ |T \cap K| = i+1}} y_T \right)$$

from which results a recurrence relation in i leading to

$$\sum_{\substack{T\\|T\cap K|=i+1}} y_T = -\frac{t-i}{i+1}\sum_{\substack{T\\|T\cap K|=i}} y_T$$
$$\vdots$$
$$= (-1)^{i+1}\binom{t}{i+1}\sum_{\substack{T\\|T\cap K|=0}} y_T,$$

and for $i+1=t$,

$$\sum_{\substack{T\\T\subset K}} y_T = (-1)^t \sum_{\substack{T\\|T\cap K|=0}} y_T,$$

which is (c). Moreover, (d) gives a linear system with $\binom{n}{t}$ variables and $\binom{n}{t-1}$ equations so the vector space of solutions has dimension $\binom{n}{t}-\binom{n}{t-1}$. Those vectors are clearly independent for distinct values of t since they correspond to the distinct $(-1)^t\binom{n-k-t}{k-t}$ eigenvalues. Therefore we have

$$1+\sum_{t=1}^{k-1}\left(\binom{n}{t}-\binom{n}{t-1}\right)=\binom{n}{k}$$

linearly independent eigenvectors. ∎

Once computed the spectrum of $KG_{n,k}$ we can then use it to prove some graph-theoretic results regarding the graphs $KG_{n,k}$ themselves as well as other merely combinatorial results. One example of the latter is the famous Theorem of Erdős, Ko and Rado on intersecting families of sets:

Theorem 4.3 (Erdős–Ko–Rado)**.** *Suppose that $\mathcal{A}$ is a family of distinct subsets of $\{1,2,\dots,n\}$ such that each subset is of size k and each pair of subsets has a nonempty intersection, and suppose that $n\geqslant 2k$. Then the number of sets in $\mathcal{A}$ is less than or equal to*

$$|\mathcal{A}|\leqslant\binom{n-1}{k-1}.$$

Proof. If we let G be the Kneser graph $KG_{n,k}$ then $\mathcal{A}\subset V(G)$ is a subset of the vertex set of G. Moreover note that $\mathcal{A}$ is actually an independent set on G, and since $KG_{n,k}$ is a regular graph of degree $d=\binom{n-k}{k}$ by using The Hoffman bound on Theorem 3.8 we have

$$|\mathcal{A}|\leqslant\frac{-n\lambda_{min}}{d-\lambda_{min}},$$

being λ_{min} the least eigenvalue of $KG_{n,r}$. By proposition 4.2 we know that $\lambda_{min}=-\binom{n-k-1}{k-1}$, hence

$$|\mathcal{A}|\leqslant\frac{\binom{n}{k}\binom{n-k-1}{k-1}}{\binom{n-k}{k}+\binom{n-k-1}{k-1}}=\binom{n-1}{k-1}.$$

∎

Note that the number of k-subsets of the n set containing a specified element i is equal to $\binom{n-1}{k-1}$, so such a family is maximal and the bound is tight. Erdős, Ko and Rado proved the Theorem back in 1938 using a common combinatorial argument known as double counting. On the other hand the proof we just gave, although less elemental, comes from a more powerful and general result regarding spectral graph theory, showing yet again the utility of this field.

As in last section, we shall now move on to the study of the Lovász number and Shannon capacity of the Kneser graphs. The following proposition will be of utility:

Proposition 4.4. *The Kneser graph $KG_{n,k}$ is a vertex-transitive and an edge transitive graph.*

Proof. Let S_n be the n-symmetric group regarded as permutations of $\{1, \dots, n\}$, and let A and B be two vertices of $KG_{n,k}$, regarded as k-subsets of $\{1, \dots, n\}$. Clearly S_n is a subgroup of the automorphism group $\Gamma(KG_{n,k})$ since for any $\pi \in S_n$ we have $A \cap B = \emptyset$ if and only if $\pi(A) \cap \pi(B) = \emptyset$. Also, for any A and B there is a $\sigma \in S_n$ such that $\sigma(A) = B$ so S_n acts transitively on the k-subsets of $\{1, \dots, n\}$ and hence $KG_{n,k}$ is vertex-transitive.

To see that is also edge-transitive let (A, A') and (B, B') be two edges of the Kneser graph and let $A = \{a_1, \dots, a_k\}$, $A' = \{a_{k+1}, \dots, a_{2k}\}$, $B = \{b_1, \dots, b_k\}$ and $B' = \{b_{k+1}, \dots, b_{2k}\}$ be their respective elements. We construct a permutation $\pi \in S_n$ by defining $\pi(a_i) = b_i$ for $i = 1, \dots, 2k$. It is well defined since we have $A \cap A' = \emptyset$, $B \cap B' = \emptyset$ and also $2k \leqslant n$. Also π verifies

$$\pi(A) = B, \quad \pi(A') = B'$$

so we conclude that $KG_{n,k}$ is also edge-transitive. ■

Actually, using the Erdos-Ko-Rado Theorem it can be shown that the automorphism group of $KG_{n,k}$ is in fact S_n. If we let $\mathcal{A}_i$ denote the maximum independent set consisting of all the k-subsets of $\{1, \dots, n\}$ which contain the element i, then any automorphism of $KG_{n,k}$ must permute its maximum independent sets, and by Theorem 4.3, all the maximum independent sets are of the form $\mathcal{A}_i$. Thus any automorphism of $KG_{n,k}$ permutes the $\mathcal{A}_i$, and therefore determines a permutation in S_n. Since any permutation other than the identity can not fix all the $\mathcal{A}_i$ it results that $\Gamma(KG_{n,k}) \cong S_n$.

We now move on to determine the Shannon capacity of the Kneser graphs. We state it in the form of the following theorem:

Theorem 4.5. *The Shannon capacity of the Kneser graph $KG_{n,k}$ is given by*

$$\Theta(KG_{n,k}) = \binom{n-1}{k-1}$$

Proof. We start again by noting that if $\mathcal{A}$ is a family of k-subsets of $\{1, \dots, n\}$ such that any $A \in \mathcal{A}$ contains a given element (say i) then $|\mathcal{A}| = \binom{n-1}{k-1}$. Also, since any pair of sets in $\mathcal{A}$ are intersecting (i belongs to the two), by regarding $\mathcal{A}$ as a subset of the vertices of $KG_{n,k}$ and by Theorem 4.3 we have

$$\alpha(KG_{n,k}) = |\mathcal{A}| = \binom{n-1}{k-1}.$$

On the other hand, $KG_{n,k}$ is an edge-transitive graph which is regular of degree $d = \binom{n-k}{k}$ and has as least eigenvalue $\lambda_{min} = -\binom{n-k-1}{k-1}$. Hence by Theorem 3.12 of the previous section we can compute the actual value of $\vartheta(KG_{n.k})$, i.e

$$\vartheta(KG_{n,k}) = \frac{\binom{n}{k}\binom{n-k-1}{k-1}}{\binom{n-k}{k} + \binom{n-k-1}{k-1}} = \binom{n-1}{k-1},$$

and using yet again that $\alpha(G) \leqslant \Theta(G) \leqslant \vartheta(G)$, the results follows. ∎

Another graph parameter known for the Kneser graphs is their chromatic number. Knesser conjectured that the chromatic number of $KG_{n,k}$ was equal to $\chi(KG_{n,k}) = n - 2k + 2$. The first proof was given by Lovász and it can be found on [5]. The proof introduced topological methods related to graph theory, and though as interesting the proof is, it is beyong the scope of this thesis.

5. Perfect graphs

In this section we will provide yet another family of graphs, the perfect graphs, for which the Shannon capacity is known and is again equal to the Lovász number. We will start by defining some convex sets on the euclidean space where both the independency number and the Lovász number arise from an optimization problem over them. Moreover, this sets also provide a geometric view of the structure of the graph. In adition to computing the Shannon capacity, the Lovász number is of special interest for the family since it also allows to compute the four graph parameters introduced in the first section, namely the clique number, the independency number, the chromatic number and the clique cover number. This follows from the Sandwich inequality that will be proved below.

Recall from the first section that a graph G is said to be *perfect* if and only if every induced subgraph H of G satisfies $\omega(H) = \chi(H)$. That is, every induced subgraph can be coloured with exactly as few colours as the size of its largest clique. Perfect graphs where first studied by Berge [6], and are notable the two conjectures himself made: *"A graph is perfect if and only if its complementary is also perfect"*, as well as *"A graph is perfect if and only if it does not contain neither an odd cycle nor the complement of an odd cycle as induced subgraphs"*. In the last part of the section we will give a proof of the first and briefly comment the second.

Examples of perfect graphs include *interval graphs*, where each vertex i of G corresponds to an interval of real numbers $I_i = [a_i, b_i] \subset \mathbf{R}$ and vertices i and j are adjacent whenever $I_i \cap I_j \neq \emptyset$. The proof of the perfection of this family of graphs was given in the first section by efectively giving a colouring of G using $\omega(G)$ colours.

A more trivial example of perfect graphs are *bipartite graphs*, where the vertex set of B can be partitioned into two disjoint sets X, Y in such a way that every edge of B has one end in X and the other end in Y. Since no pair of vertices either both in X or both in Y are adjacent, two colours are suficient to colour B (one for X and another for Y). Also every edge is itself a maximum clique of B, so $\omega(B) = \chi(B) = 2$ and thus B is perfect.

The complement of a bipartite graph is also a perfect graph. If B is a bipartite graph and $G = \overline{B}$, then every induced subgraph of G is the complement of an appropriate subgraph of B, and thus also bipartite, so it suffices to show that $\omega(G) = \chi(G)$. This equality in terms of the complement reads $\alpha(B) = \overline{\chi}(B)$ and is in fact true due to a classical result of König [7] on graph theory, stating that the vertices of any bipartite graph B can be covered by $\alpha(G)$ edges and vertices (i.e. by cliques of size 1 or 2).

Finally, for an example of a family of non perfect graphs, take the cursed odd cycles. Clearly all cycles C_n satisfy $\omega(C_n) = 2$. For n odd however, if C_n is on vertices $\{1, \dots n\}$ and consecutive vertices are adjacent, then odd numbers have to be coloured different than the even numbers, so at least two colours are needed. Since 1 and n are both odd and also adjacent we conclude that $\chi(C_n) = 3$.

5.1 Convex labelings

Definition 5.1 (Convex set, convex hull)**.** A subset $U \subset \mathbf{R}^d$ of the usual euclidean space is said to be a *convex set* if for any two points $x, y \in U$, then $tx + (1-t)y \in U$ for $0 \leqslant t \leqslant 1$. That is, given any two points on the set, the line segment connecting them also belongs to the set.

Given a subset $S \subset \mathbf{R}^d$, the *convex hull* of S is the smallest convex set which contains S.

Definition 5.2 (Real labelings)**.** Let G be a graph on vertices $V = \{1, \dots, n\}$. A *real labeling* of G is an assignement $\mathbf{x}$ of real numbers x_i to each vertex $i \in V$.

If $U \subset V$ is a subset of the vertex set of G, the *characteristic labeling* for U is defined to be $\mathbf{x}_U = (x_1, \dots, x_n)$ where

$$x_i = \begin{cases} 1 & \text{if } i \in U \\ 0 & \text{if } i \notin U \end{cases}$$

A characteristic labeling for a stable set of vertices is called a *stable labeling*, and a characteristic labeling for a clique is called a *clique labeling*. We define

$$\mathrm{STAB}(G) = \text{convex hull}\,\{\mathbf{x} \in \mathbf{R}^n : \mathbf{x} \text{ is a stable labeling of } G\},$$

$$\mathrm{TH}(G) = \left\{\mathbf{x} \in \mathbf{R}^n_+ : \sum_{i \in V} (\mathbf{c}^\mathsf{T}\mathbf{u}_i)^2 x_i \leqslant 1 \;\; \begin{array}{l} \text{for all ortonormal representation } (\mathbf{u}_1, \dots, \mathbf{u}_n) \text{ of} \\ G \text{ and for all } \mathbf{c} \in \mathbf{R}^n \text{ with } \mathbf{c}^\mathsf{T}\mathbf{c} = 1. \end{array}\right\},$$

$$\mathrm{QSTAB}(G) = \left\{\mathbf{x} \in \mathbf{R}^n_+ : \sum_{i \in Q} x_i \leqslant 1 \text{ for all cliques } Q \text{ of } G \right\}.$$

For any of the three sets $\mathrm{STAB}(G)$, $\mathrm{QSTAB}(G)$ and $\mathrm{TH}(G)$ of a given graph G on vertices V we can define three graph parameters as follows:

$$\alpha_1(G) = \max\left\{\sum_{i \in V} x_i : \mathbf{x} \in \mathrm{STAB}(G)\right\},$$

$$\vartheta_1(G) = \max\left\{\sum_{i \in V} x_i : \mathbf{x} \in \mathrm{TH}(G)\right\},$$

$$\kappa(G) = \max\left\{\sum_{i \in V} x_i : \mathbf{x} \in \mathrm{QSTAB}(G)\right\}.$$

Proposition 5.3. *The numbers $\alpha_1(G)$ and $\vartheta_1(G)$ are the independency number $\alpha(G)$ and the Lovász number $\vartheta(G)$ of G, respectively.*

Proof. Let V be the vertex set of G, let $S \subset V$ be a maximum independent set with $|S| = \omega(G)$ and let $\mathbf{x} \in STAB(G)$ be the characteristic labeling for S (i.e. $x_i = 1$ if $i \in S$, $x_i = 0$ otherwise). Then

$$\alpha(G) = \sum_{i \in V} x_i \leqslant \max\left\{\sum_{i \in V} x_i : \mathbf{x} \in \mathrm{QSTAB}(G)\right\} = \alpha_1(G).$$

For the reverse inequality, let $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ be the set of all stable labelings of G. Any vector $\mathbf{x} \in \mathrm{STAB}(G)$ is a convex combination of the $\mathbf{x}_1, \dots, \mathbf{x}_m$ so

$$\mathbf{x} = \sum_{j=1}^{m} a_j \mathbf{x}_j$$

for some $a_j \geqslant 0$ with $a_1 + \cdots + a_m = 1$. If we let $\mathbf{x}$ be the vector in $STAB(G)$ attaining the maximum value for $\sum_{i \in V} x_i$ we have:

$$\alpha_1(G) = \sum_{i \in V} x_i = \sum_{i \in V} \sum_{j=1}^{m} a_j (x_j)_i = \sum_{j=1}^{m} a_j \sum_{i \in V} (x_j)_i \leqslant \sum_{j=1}^{m} \alpha(G) a_j = \alpha(G)$$

since all $\mathbf{x}_j$ satisfies $\sum_{i \in V} (x_j)_i \leqslant \alpha(G)$. Therefore $\alpha_1(G) = \alpha(G)$.

For the Lovász number, let $\mathbf{x} \in \mathrm{TH}(G)$ be the vector which maximizes $\sum_{i \in V} x_i$ and let $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ be an optimal orthonormal representation for G with handle $\mathbf{c}$. Then

$$\vartheta_1(G) = \sum_{i \in V} x_i \leqslant \left(\max_{i \in V} \frac{1}{(\mathbf{c}^\mathsf{T} \mathbf{u}_i)^2} \right) \sum_{i \in V} (\mathbf{c}^\mathsf{T} \mathbf{u}_i)^2 x_i \leqslant \max_{i \in V} \frac{1}{(\mathbf{c}^\mathsf{T} \mathbf{u}_i)^2} = \vartheta(G).$$

The reverse inequality requires some advanced techniques on polyhedral optimization, which can be found on [8]. ■

The set $\mathrm{QSTAB}(G)$ was defined by Shannon on [1] in order to find bounds on the Shannon capacity of a graph. A systematic study of $\mathrm{STAB}(G)$ and $\mathrm{TH}(G)$ by Grötschel, Lovász and Schrijver can be found in [8] and [9]. One of the motivations for this was that as last proposition shows, the independency number can be computed by optimizing an integral linear program on the convex set $\mathrm{STAB}(G)$. However, the inequalities needed to characterize the convex set $\mathrm{STAB}(G)$ are not fully known in general. Thus, the problem is relaxed to allow non-integral solutions and considered over the broader (as we will see next) sets $\mathrm{TH}(G)$ and $\mathrm{STAB}(G)$ with hope to recover then a solution for the original problem.

5.2 Sandwich Theorem

We start by formalizing what we meant by broader before.

Proposition 5.4. *The set* $\mathrm{TH}(G)$ *is sandwiched between* $\mathrm{STAB}(G)$ *and* $\mathrm{QSTAB}(G)$*:*

$$\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{QSTAB}(G)$$

Proof. Let $\mathbf{x}$ be a stable labeling for an independent set S. Then for any unitary $\mathbf{c}$ and any orthonormal representation $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ of G we have:

$$\sum_{i \in V} (\mathbf{c}^T \mathbf{u}_i)^2 x_i = \sum_{i \in S} (\mathbf{c}^T \mathbf{u}_i)^2 \leqslant \mathbf{c}^\mathsf{T} \mathbf{c} = 1$$

and since $\mathrm{TH}(G)$ is a convex set, it then contains the convex hull of all stable labelings of G. Hence $\mathrm{STAB}(G) \subset \mathrm{TH}(G)$.

On the other hand, note that every clique labeling of G is also trivially an orthonormal representation of G of dimension one. Therefore, if we let $c = 1$ and $(u_1, \ldots, u_n)$ be a clique labeling for some clique Q, then for all $\mathbf{x} \in \mathrm{TH}(G)$ we have

$$1 \geqslant \sum_{i \in V} (c \cdot u_i)^2 x_i = \sum_{i \in Q} x_i$$

and thus $\mathbf{x} \in \mathrm{QSTAB}(G)$. ■

Theorem 5.5 (Sandwich inequality). *For any graph G we have*

$$\alpha(G) \leqslant \vartheta(G) \leqslant \kappa(G) \leqslant \chi(\overline{G}).$$

Proof. Propositions 5.3 and 5.4 establish the already known $\alpha(G) \leqslant \vartheta(G)$, as well as $\vartheta(G) \leqslant \kappa(G)$.

To see $\kappa(G) \leqslant \chi(\overline{G})$, note that by colouring $\overline{G}$ with $k = \chi(\overline{G})$ colours, each of the colour classes of $\overline{G}$ induces a partition of the vertices of G into k dijoint cliques $Q_1, \dots, Q_k$ (one for each colour class). Suppose $\mathbf{x} \in \mathrm{QSTAB}(G)$ attains the maximum $\kappa(G)$. Then

$$\kappa(G) = \sum_{i \in V} x_i = \sum_{i \in Q_1} x_i + \cdots + \sum_{i \in Q_k} x_i \leqslant 1 + \cdots + 1 = k,$$

so $\kappa(G) \leqslant \chi(\overline{G})$. ■

The major relevance of Theorem 5.5 becomes clear if we apply it to the complement of G. Using $\alpha(\overline{G}) = \omega(G)$ we get

$$\omega(G) \leqslant \vartheta(\overline{G}) \leqslant \chi(G). \tag{5.1}$$

Both the clique number and the chromatic number of a graph are hard numbers to compute in general (no known polynomial algorithm is known), but the Lovász number $\vartheta(G)$ is not. In fact, it can be shown that $\vartheta(G)$ can be computed from a semidefinite optimization program ([10] for greater detail), and thus in polynomial time. Since both $\omega(G)$ and $\chi(G)$ are integeers, computing an aproximation of $\vartheta(\overline{G})$ and rounding the result to the nearest integeer gives a good aproximation on both $\omega(G)$ and $\chi(G)$.

If in addition the graph G is a perfect graph, then inequalites 5.1 become an equality since $\omega(G) = \chi(G)$, and thus $\vartheta(\overline{G})$ completely determines both the clique number and the chromatic number of G. Therefore we have the following Theorem:

Theorem 5.6. *The Shannon capacity of a perfect graph is equal to its Lovász number.*

Proof. From the definition of the Shannon capacity and from Theorem 3.8

$$\alpha(G) \leqslant \Theta(G) \leqslant \vartheta(G)$$

but using equation 5.1, if G is a perfect graph then $\omega(G) = \vartheta(\overline{G})$, or equivalently $\alpha(G) = \vartheta(G)$ and therefore $\Theta(G) = \vartheta(G)$. ■

5.3 The weak perfect graph Theorem

We are ready to give a proof for the first conjecture of by Berge stating that a graph is perfect if and only if its complementary is perfect too. The first proof was due to Lovász who used only combinatorial arguments. However, here we take a more geometrical aproach using facts about the polytopes STAB and QSTAB introduced above.

We start by a simple characterization of perfect graphs that will prove very useful. We state it in the form of the following lemma.

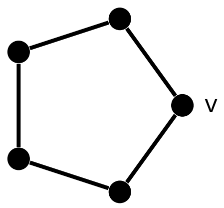
Lemma 5.7. *A graph G is perfect if and only if every induced subgraph H of G has an independent set that intersects every maximum clique of H.*

Proof. Suppose first that G is perfect and let H be an induced subraph H of G (which is perfect too), and colour H with $\chi(H)$ colours including red. Then the set S consisting of all red vertices forms an independent set of H, and since every maximum clique Q of H contains a red vertex, S intersects Q.

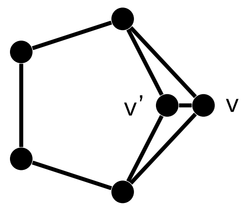
For the converse, let H be an induced subgraph of G. We will proceed by induction on the size of the largest clique of H. For $\omega(H) = 1$ we have that H is trivially perfect. Now suppose that H is perfect for all H satisfying $\omega(H) \leqslant k-1$ and let H' be any induced subgraph of G such that $\omega(H') = k$. From the hypothesis of the lemma we have that H' has a subset S of the vertices that intersects every maximum clique of H'. Then the graph $H' \backslash S$ formed by removing every vertex in S from H' has maximum cliques of size at most $k-1$, so $H' \backslash S$ is perfect and therefore it can be coloured with $k-1$ colours. Assigning the remaining colour k to the set S results in a colouring of H' with k colours, which implies that $\chi(H') = \omega(H')$ and by the induction principle results that G is perfect. ■

The following definition was introduced by Lovász in his argument for the proof of the weak perfect graph Theorem.

Definition 5.8 (Vertex expansion)**.** Let G be a graph on vertices V and let $v \in V$ be a vertex. We define the graph G^{v+} as the graph obtained by adding a new vertex v' to G, an edge vv' and edges uv' for all $u \in V$ adjacent to v in G. The resulting graph is G with the vertex v *expanded*.



(a) C_5 graph



(b) vertex expanded graph C_5^{v+}

Lemma 5.9. *If G is perfect, so is G^{v+} for any vertex $v \in V(G)$.*

Proof. First of all we note that any induced subgraph of G^{v+} is either an induced subgraph of G (if it omits v or v' or both) or is of the form H^{v+} for some induced subgraph H of G (if it retains both v and v'). Therefore by lemma 5.7 it suffices to show that G^{v+} itself has an independent set that intersects every maximum clique of G^{v+} because the same argument then applies to all induced subgraphs of G^{v+} of the form H^{v+}.

Since G is perfect, we start by colouring G with $\omega(G)$ colours, and say vertex v recieves the colour red. Let S be the independent set consisting of all red vertices of G. If Q is a maximum clique of G^{v+} not containing v' then Q is also a maximum clique of G, and thus contains a red vertex (possibly v) implying that S intersects Q. On the other hand, if Q contains v' then it must also contain v since otherwise $Q \cup \{v\}$ would be a clique of G^{v+} larger than Q. Therefore Q contains the red vertex v so S intersects Q too and thus we conclude that G^{v+} is perfect. ■

The following two propositions are the heart of the proof of the weak perfect graph Theorem, altogether with an other characterization of perfection for a graph regarding the sets $\mathrm{STAB}(G)$ and $\mathrm{QSTAB}(G)$.

Proposition 5.10. *If G is a perfect graph, then* $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$.

Proof. We already know $\mathrm{STAB}(G) \subseteq \mathrm{QSTAB}(G)$ from proposition 5.4. To show $\mathrm{QSTAB}(G) \subseteq \mathrm{STAB}(G)$, since $\mathrm{STAB}(G)$ is a closed set, it suffices to show that $\mathbf{x} \in \mathrm{QSTAB}(G)$ implies $\mathbf{x} \in \mathrm{STAB}(G)$ only for vectors $\mathbf{x}$ with rational coordinates.

Let G have vertices $\{1, \dots, n\}$ and suppose $\mathbf{x} = (x^1, \dots, x^n) \in \mathrm{QSTAB}(G)$ is a vector with rational coordinates. Then there exists some integeer q for which $q\mathbf{x}$ has integeer coordinates. Now let G^+ be the graph obtained by repeteadly expanding its vertices until each original vertex i has been replaced by a clique of size qx_i.

From the definition of $\mathrm{QSTAB}(G)$, for any clique Q of G we have

$$\sum_{i \in Q} x_i \leqslant 1.$$

Now every clique Q' of G^+ is contained in a clique of size $\sum_{i \in Q} qx_i$ for some clique Q of G. Thus $\omega(G^+) \leqslant q$, and since $\omega(G^+)$ is perfect by lemma 5.9, we can colour $\omega(G^+)$ with q colours.

For each colour k, $1 \leqslant k \leqslant q$, let $\mathbf{x}_k$ be the stable labeling defined by $(x_k)_i = 1$ if some expanded vertice of i is coloured k, $(x_k)_i = 0$ otherwise. Then

$$\frac{1}{q} \sum_{k=1}^{q} \mathbf{x}_k \in \mathrm{STAB}(G),$$

but since very vertex of G^+ is coloured, for all $i \in V$ we have

$$\sum_{k=1}^{q} (x_k)_i = qx_i.$$

Hence $\mathbf{x} = \frac{1}{q} \sum_{k=1}^{q} \mathbf{x}_k$ and $\mathbf{x} \in \mathrm{STAB}(G)$. ∎

Proposition 5.11. *If a graph G satisfies $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$, then its complementary $\overline{G}$ is perfect.*

Proof. We start by showing that if $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ then also $\mathrm{STAB}(H) = \mathrm{QSTAB}(H)$ for all induced subgraphs $H \subset G$. Let H be an induced subgraph of G and let $\mathbf{x} \in \mathrm{QSTAB}(H)$ be a real labeling for H. We extend $\mathbf{x}$ to a real labeling $\tilde{\mathbf{x}}$ of G by seting $\tilde{x}_i = 0$ for all vertices i of $G \backslash H$. Clearly

$$\tilde{\mathbf{x}} \in \mathrm{QSTAB}(G)$$

and because $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ we have $\tilde{\mathbf{x}} \in \mathrm{STAB}(G)$. This itself implies that $\tilde{\mathbf{x}}$ is a convex combination of all stable labelings $\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_m$ of G, i.e

$$\tilde{\mathbf{x}} = \sum_{j=1}^{m} a_j \tilde{\mathbf{x}}_j$$

for some $a_j \geqslant 0$ with $a_1 + \cdots + a_m = 1$. Now note that if $\tilde{\mathbf{x}}_j$ is a stable labeling for an independent set containing vertices that are not in H, it must be $a_j = 0$. Therefore by suppressing on vectors $\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_m$ all coordinates $(\tilde{x}_j)_i$ referring to vertices i not in H and by removing vectors $\tilde{\mathbf{x}}_j$ with $a_j = 0$ we get a set of stable labelings $\mathbf{x}_1, \dots \mathbf{x}_{m'}$ of H satisfying

$$\mathbf{x} = \sum_{j=1}^{m'} a_j \mathbf{x}_j.$$

Hence $\mathbf{x} \in \mathrm{STAB}(H)$.

Now by applying lemma 5.7 on the complementary of G we have that $\overline{G}$ is perfect if and only if every induced subgraph H of G has a clique that intersects every maximum independent set of H. Using this fact and the fact proved above that $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ is inherited by induced subgraphs, it suffices to show that G has a clique that meets every maximum stable set of G.

We prove this assertion by induction on $|V(G)|$, the number of vertices in G. For $|V(G)| = 1$ this is trivially true. Suppose also that is true for all G with $|V(G)| \leqslant k-1$ and let G have $|V(G)| = k$. We may assume that any vertex of G belong to a maximum independent set, for otherwise we may delete such vertex and get a G with $|V(G)| < k$.

Let X be the subset of $\mathrm{STAB}(G)$ consisting of all vectors $\mathbf{x} = (x_1, \dots x_n)$ satisfying

$$\sum_{i \in V} x_i = \alpha(G).$$

Clearly X is a facet of $\mathrm{STAB}(G)$, so it must be also a facet of $\mathrm{QSTAB}(G)$. Therefore for any vector $\mathbf{x} \in X$ one of the following two inequalities defining $\mathrm{QSTAB}(G)$ must be an equality:

$$x_i \geqslant 0 \quad \text{for every } i \in V. \tag{a}$$

$$\sum_{i \in Q} x_i \leqslant 1 \quad \text{for every clique } Q \text{ of } G. \tag{b}$$

Since every vertex of G belongs to a maximum independent set, it follows that none of the inequalities (a) can be an equality for all $\mathbf{x} \in X$, and hence one of the inequalities (b) is. Thus there exists a clique Q of G which meets every maximum independent set, as desired. ∎

Theorem 5.12 (Weak perfect graph Theorem)**.** *A graph G is perfect if and only if its complementary $\overline{G}$ is perfect.*

Proof. Propositions 5.10 and 5.11 establish that if G is perfect then $\overline{G}$ is perfect too. Applying the same results to $\overline{G}$ we get the converse. ∎

Actually we have the following characterization of perfection:

Corollary 5.13. *For any graph G, the following four conditions are equivalent.*

(i) *G is perfect.*

(ii) $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$.

(iii) *$\overline{G}$ is perfect.*

(iv) $\mathrm{STAB}(\overline{G}) = \mathrm{QSTAB}(\overline{G})$. ∎

Theorem 5.14 (Chudnovsky, Robertson, Seymour, Thomas)**.** *A graph is perfect if and only if it does not contain neither a odd cycle (an odd hole) nor the complement of an odd cycle (an odd antihole) as induced subgraphs.*

Clearly Theorem 5.14 implies the weak perfect graph Theorem on 5.12 since for any forbidden induced subgraph, its complementary is also forbidden. This Theorem was proven in a paper appearing in the Annals of Mathematics in 2006, and won the 2009 Fulkerson Prize and a cash award of $10,000. Its remarkable to note that the complete proof contains more than 170 pages.

6. Conclusion

To recapitulate, we shall summarize the goals met by this thesis. In section 1 we have started by introducing all the notions of the field of Graph Theory required to formulate and understand the problem of Shannon regarding the maximum rate at which information can be transmitted through a channel of communication, also known as the Shannon capacity. In the way we have defined four graph parameters of major interest in the field: the independency number, the clique number, the chromatic number and the clique covering number. In addition, we have defined the concept of strong product of graphs and provided an explicit computation of the Shannon capacity for the bowtie graph.

We have then explored the field of Spectral Graph Theory in section 2 to approach the problem from an algebraic point of view, defining the spectrum of a graph and using it to give bounds to the independency number and clique number of regular graphs. In addition, we have determined the spectrum of the strong product of graphs in terms of the spectrum of each component, and we have effectively computed the spectrum of the family of cycle graphs.

With the Shannon's problem in mind, in section 3 we have introduced the Lovász number of a graph and proved that it serves an upper bound to the Shannon capacity. This number has allowed us to solve the problem of Shannon for the case of the Pentagon (which remained unsolved until Lovász) in two different ways: one by the original argument given by Lovász on his article, and another by deriving a spectral bound for the Lovász number equivalent to the one given on section 2. Then we have proved this bound to be equal to the Lovász number of a graph for all edge-symmetric graphs and used this result to compute the capacity of such a family of graphs as they are the even cycles. For Odd cycles however the Lovász number is not a good enough bound to actually compute their capacity, so only an upper bound can be given.

Then, in section 4 we have defined another family of edge-symmetric graphs, the Kneser graphs, which arise as the intersection structure of some subsets of integers. In a similar fashion to section 3 we have computed their spectrum to determine their Lovász number and then used this result to compute the Shannon capacity for the family. In addition, we have proved a famous Theorem by Erdős, Ko and Rado using the spectral bound on the independency number found on section 2.

In the last section we have given yet another family of graphs for which the Lovász number also determines the Shannon capacity. Those graphs are the perfect graphs, introduced by Berge also in an attempt to study the Shannon capacity. We have proved that the Lovász number of any graph always resides between the independency number and the clique covering number of the graph. We have then used this result (known as the Sandwich inequality) for the case when the graph in question is perfect to show how the Lovász number fully determines the four graph parameters introduced in section 1, as well as the Shannon capacity of the graph. Finally we have concluded this section by giving a proof for the weak perfect graph conjecture and by stating the strong perfect graph conjecture. Both are now Theorems, and the second fully characterizes when a graph is perfect.

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